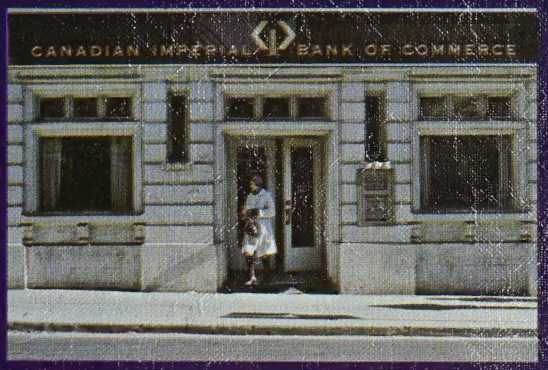
MATH PROBE

3

BYE ELLIOTT



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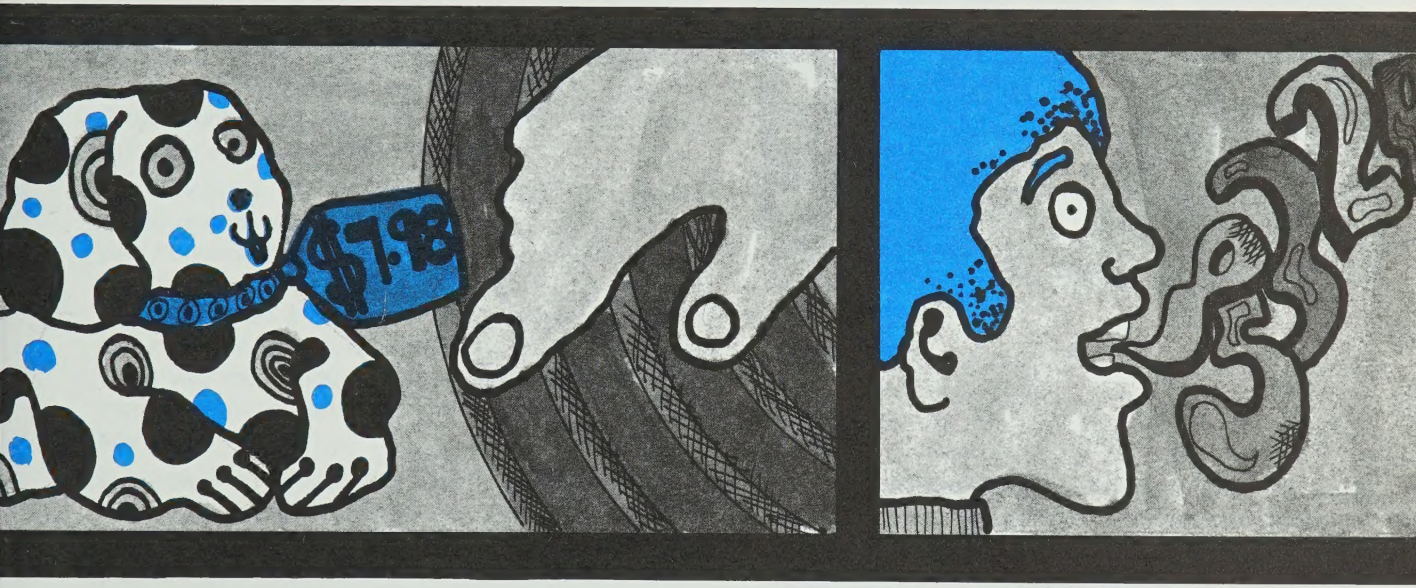

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Math Probe 3

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A Note From the Authors

To The Student:

This book is written to you – the student. A number of questions are asked just as if we were talking to you – and often no answers are given – immediately. You should think about the questions, try to answer them, discuss them, then read on.

There are several features of note in this book. It is intended that the Projects be done by groups of two to five students working cooperatively, each doing his part, then sharing the results. In most instances the Projects are to provide experiences upon which to build the formal concepts later. In some cases you will be able to do the exercises without further help. One important point: you can learn much by discussing the Projects with each other and your teacher. The discussions are a vital part of this program.

The Discovery Exercises are designed primarily for you to work by yourself though you may wish to check your conclusions with your classmates. While mathematics at times is something you must do alone, there are times when it is more efficient, more interesting, and more fun to work with others. But there are times when you must practise alone. Mathematics is like most sports – you only become good at it when you practise.

No book is complete within itself. This is true of this book. In a number of instances it is suggested the class write letters to obtain information, booklets, and publications. This should be done ahead of time to ensure the material is on hand when it is needed. Also it is suggested you invite people from your community into your classroom to discuss certain topics. At other times it is suggested that you visit the community and business world. In these ways mathematics is interwoven with the community and with meaningful aspects of living – it becomes relevant.

"Using the Library" is included in each chapter to encourage you to look at the history of mathematics, at some interesting applications, and at some of the interesting people who contributed to mathematics. Suggestion: look ahead – start the assignment early. Don't wait until the end of the chapter to start using the library.

Some topics in this book were developed in **Math Probe 2.** The presentation of these topics in this book is sufficiently different that you will find the material somewhat familiar while adding a new dimension to the topics. If you did not cover these topics before you may want to refer to **Book 2** for some interesting aspects and developments. The student who covered the topics may wish to move quickly through the material doing only those exercises relating to the concepts forgotten or not covered. The projects and activities are related to those in **Book 2** yet are different. You may want to do some of the projects outlined in **Book 2**.

Included are actual problems from industry. It is intended that these only be a beginning. Your school may wish to contact employers in your area for more examples of mathematics and your future work.

We hope you will enjoy probing mathematics.

M.P.B.

H.A.E.

2515240

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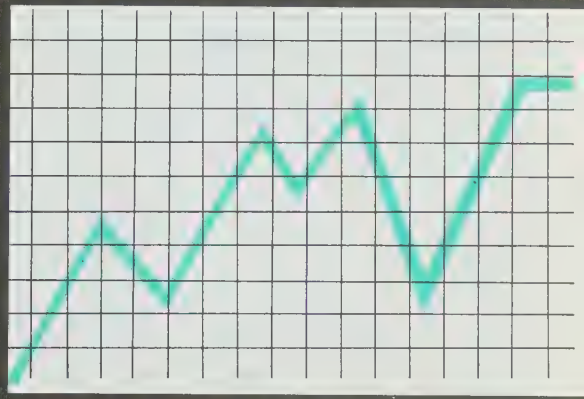
Chapter One
GRAPHS



1·1 Using Graphs

Project

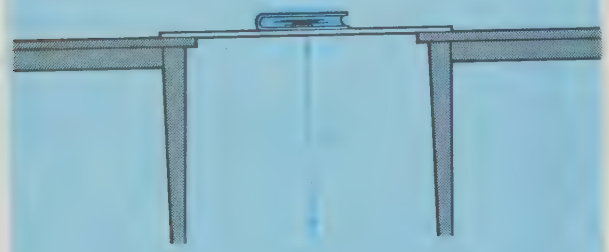
Project 1

Problem

How does a board which is suspended between two blocks bend when unit weights are placed on the middle of it?

Method

1. Obtain a slat (about 2 inches by $\frac{1}{4}$ inch by 4 feet).
2. Set up as in the diagram.



3. Record height of slat above floor.
4. Place one book in the middle of the slat. Record height above the floor. Record the sag.
5. Repeat with 2 books, 3 books, etc. (Do not place too many books on the slat.)
6. Record the information in as many ways as possible.
7. Draw as many different graphs as you can.
8. Answer questions 1 through 7 below, then proceed with Step 9.
9. Obtain 1000-gram or 1-pound weights from the science lab. Repeat Steps 1 through 5 above, but attach weights one at a time by means of a wire, in place of the books.
10. Record the information from Step 9 by a graph.
11. Repeat Step 9 but use a slat 2 feet long. (Just use $\frac{1}{2}$ the length of the original slat.) Draw a graph using the data.

Observations and Questions

1. Which of your methods of recording the information (Step 6) is the easiest to read and understand?
2. Which graph (7) tells the story the best?
3. Did the graph that shows the height above the floor or the graph that shows the amount of sag (or bend) tell you the story the best? Why?
4. Are your graphs continuous lines or points on the coordinate plane? Why?
5. Can you predict from your graph the amount of bend caused by 15 books? Why?
6. Is it important to use the same size of book each time one is added to the slat? Why?
7. If a one-pound unit was put on the slat instead of a book each time, would it be reasonable to draw a continuous graph? Why?
8. How does the amount of bend (sag) in Step 9 compare to the sag in Step 11?
9. How do you think the thickness of the slat affects the amount of sag? Can you devise an experiment to check your answer?
10. How do you think the width of the slat affects the amount of sag?

1·2 Graphing of Experimental Data

Mathematical problems occur in many forms. Some of the forms may not appear to be mathematical yet. In fact, they have mathematical relationships inherent. Let us look at the forms mathematical problems may take by studying some examples in this and the next section. The following is an outline of the forms we will study.

1. *The Problem May Occur in a Physical Situation.*

   Often when the problem appears in the physical situation no numbers are attached but numbers may be used in deriving a solution.

When a girl is rearranging the furniture in her room, she may wish to know whether the bed and bureau will fit along a certain wall. She need not know the measure of the length of the wall, nor of the bureau, nor of the bed. How many ways can you think of to solve the problem? How many of these ways involve numbers? How many do not involve numbers? Which way is the easiest?

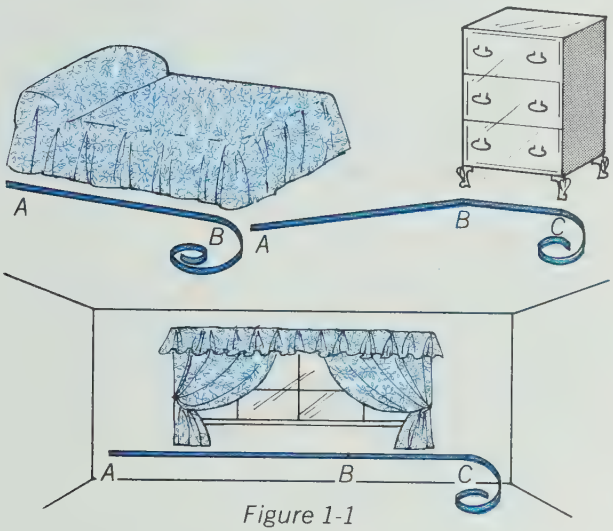


Figure 1-1

We realize that by using a measuring device, such as a yardstick or a tape measure, the whole operation is *neater.*

2. *The Problem May Occur as a Graph*

Everyone today who reads the newspaper or any magazine is confronted with graphs. Consider the graph shown in Figure 1-2. What does it tell you? What questions can you ask that can be answered directly from the graph or by some calculations?

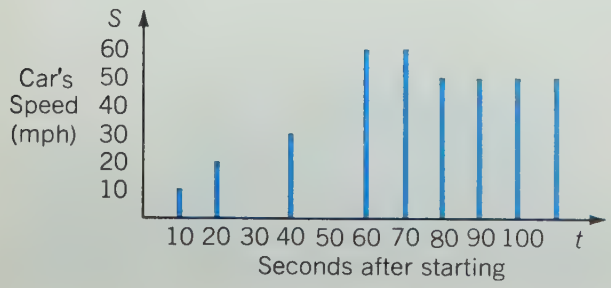


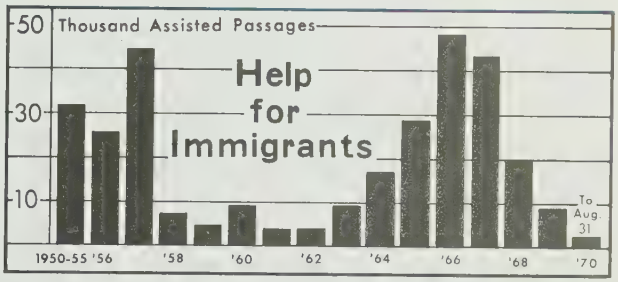
Figure 1-2

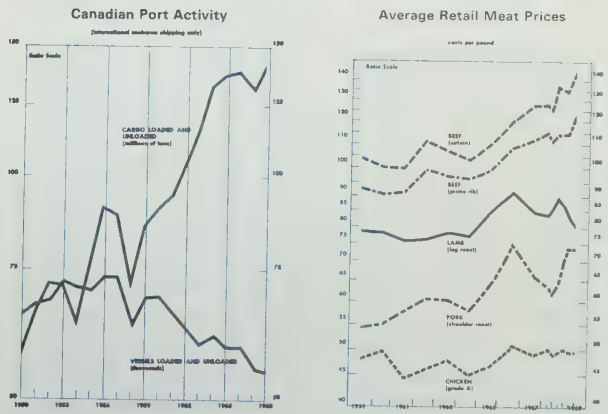
(a) What is the car's speed after 15 seconds? after 30 seconds? after 45 seconds? after 65 seconds? after 85 seconds? Can you answer these questions with certainty? Why? Which answer is likely to be most accurate? Why?

(b) Draw a smooth curve joining the endpoints. If the points on this curve represent the speed at any instant, what assumption was made about the motion of the car? Now, can you answer the questions in part (a) with any more certainty than you did previously?

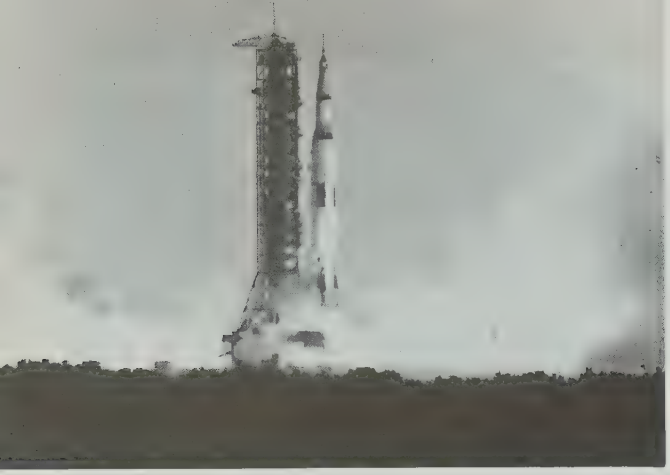
(c) Using the smooth curve find the speed of the car after 5 seconds, after 120 seconds, after 150 seconds. Can you answer these questions with any certainty? Why? What assumption did you draw when you estimated these answers?

Immigrant Loans Reduced To Trickle





Graphs are used to tell a story. Care must be taken in reading the message. Can you see any examples of misrepresentation in these illustrations? Discuss what each illustration tells you.



3. *The Problem May Appear as a Set of Ordered Pairs.*

Table 1

TIME	10	20	30	40	50	60	70	80
HEIGHT	1.1	3.3	6.1	8.4	13.9	20.5	27.1	33.7

Consider the set of ordered pairs in Table 1. The time refers to the number of seconds after blastoff and height refers to the number of miles the space vehicle is above the surface of the earth. What questions can you ask that can be answered from the data given?

(a) How far above the surface of the earth is the vehicle after 15 seconds? after 25 seconds? after 35 seconds? Can you answer these questions with any certainty? Why?

(b) How far above the surface of the earth is the vehicle after 5 seconds? after 90 seconds? after 100 seconds? Can you estimate these answers with any certainty? Why?

4. *The Problem May Involve a Formula or Equation.*

Perhaps the mathematics problem you most often think of involves a formula or an equation. For example, $A = 4\pi r^2$. No doubt you recognize this as the formula for the area of a sphere with radius r. The area is 4 times π (approximately 3.14) times the square of the radius. How many different kinds of questions can you ask using this equation? Write one question of each kind and give them to your partner to answer. You should have the answer to each question you make up.

1·3 Learning to Make the Most of Graphs

A do-it-yourself mason undertakes to build a brick wall around his backyard. He builds it in lineal sections. The table below shows the number of lineal feet completed each week.

WEEK	0	1	2	3	4	5	6	7	8	9
FEET COMPLETED	0	15	45	60	90	100	130	145	180	200

These values are plotted on a graph in Figure 1-3.

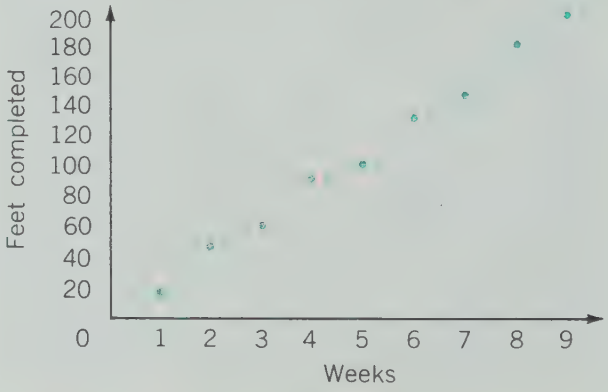


Figure 1-3

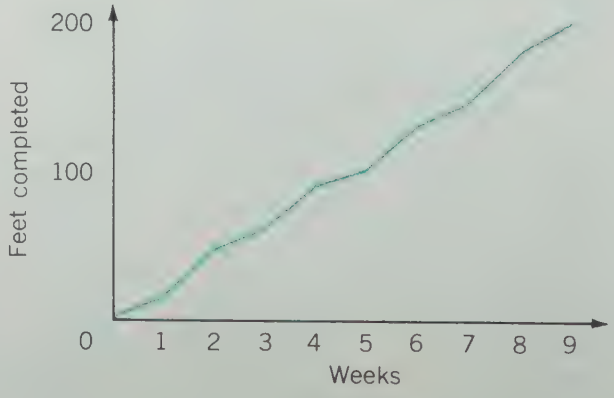


Figure 1-4

The question to ponder now is *Should the points be joined with segments as in Figure 1-4?* In drawing the segments we assume that the mason built the same

number of feet of fence each day of that week. That is, the rate of building in any one week was constant. It may be, in fact, that the mason did nothing at all the first 6 days of the first week and then built 15 feet on the 7th day. In this case, had we recorded the amount built each day, part of the graph might have looked like that in Figure 1-5.

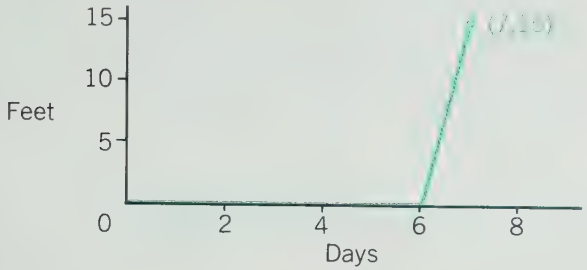


Figure 1-5

If the do-it-yourself mason had built the same number of feet of fence each day and each week the graph would have been a straight line segment joining the points (0, 0) and (9,200). We see this segment in the graph in Figure 1-6. Notice that the straight line seems to go down the *middle* of the points in this graph. How many feet per week would the mason have to build in order to produce a straight line graph?

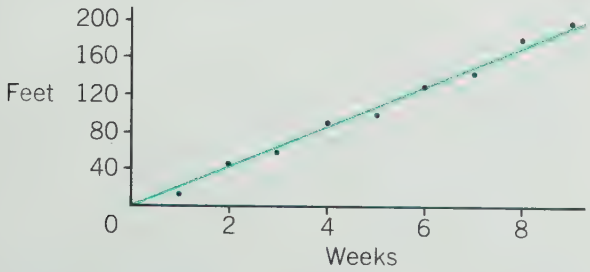
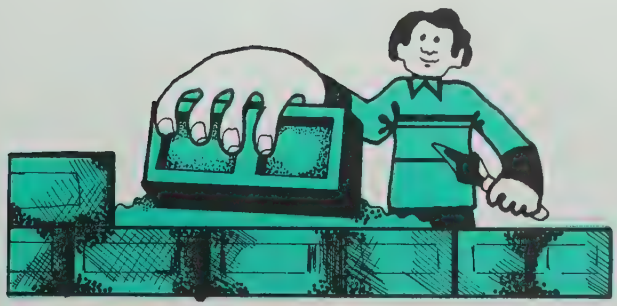


Figure 1-6



The information about the mason can be interpreted differently and a different table of data produced. Hence a different graph can be drawn. We know there was 200 feet of fence to be built, hence this table of values can be made:

AT END OF WEEK	0	1	2	3	4
FEET YET TO BE COMPLETED	200	185	155	140	110
AT END OF WEEK	5	6	7	8	9
FEET YET TO BE COMPLETED	100	70	55	20	0

This data recorded in the table appears in graphical form in Figure 1-7. The straight segment (color) indicates the average number of feet necessary to complete the fence in equal stages. Compare this with the graph in Figure 1-6.

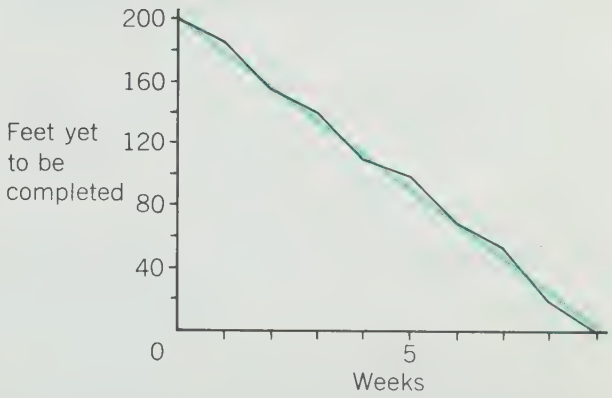


Figure 1-7

The segment in Figure 1-6 is the graph of a direct variation. It is a linear relation showing that as the number of days increase, the number of feet of fence increases directly. The colored segment in Figure 1-7 is the graph of a partial variation in that the amount of fence to be completed on any given day depends partly on the amount of fence he wanted to build initially and partly on the amount he has done in the previous weeks. Hence, since it depends only partly on the number of weeks, we say the number of feet of fence yet to be completed varies partially as the number of weeks worked.

EXERCISE 1·3

1. Tell whether each of the following pairs are related by a direct or partial variation. Be prepared to support your answer.
   (a) days worked, money earned
   (b) hours studied, percent correct on exam
   (c) money spent, money in wallet
   (d) miles travelled, hours travelled
   (e) miles travelled, distance from home
   (f) interest, interest rate
   (g) horsepower, speed of car possible
   (h) horsepower of motor, acceleration of car possible
   (i) horsepower of motor, miles per gallon of gas
   (j) length of skirt, yards of material necessary to make skirt
   (k) amount of rainfall, amount of water used for irrigation

2. Record in table form the sum of money the students in your row have. The sum for two students is the sum the first and second students in your row have; the sum for three is the sum the first three students have, etc.

(*N*) NUMBER OF STUDENTS	1	2	3	4	5	6	7
(*A*) AMOUNT							

   (a) Record the data on a graph.
   (b) Should the graph be a continuous graph? Why?
   (c) Is the change in *A* constant for each change of 1 in *N*?
   (d) Is *A* a linear function of *N*? Why?

3. Water flows into an empty uniform cylindrical tank of diameter 7 feet and depth of 10 feet at a steady rate of 3.85 cubic feet per minute. Use $\pi = 3\frac{1}{7}$.
   (a) Set up a table of values to show the depth of water *(D)* after *t* minutes.
   (b) Draw the graph relating *D* and *t*.
   (c) Should this be a continuous graph? Why?
   (d) How long does it take to fill the tank?
   (e) What is the depth of water in the tank after $3\frac{1}{2}$ minutes? after 20 minutes?
   (f) Determine the change in *D* for each change of 1 in *t*. Is this change in *D* constant?
   (g) Does *D* double when *t* doubles? Does *D* triple when *t* triples? Does *D* increase *n*-fold when *t* increases *n*-fold?
   (h) Is the graph a linear graph?
   (i) Is this relation a direct or partial variation?

1·4 Types of Graphs

We shall elaborate on the types of graphs. Refer to the graphs in Figures 1-8 through 1-11.

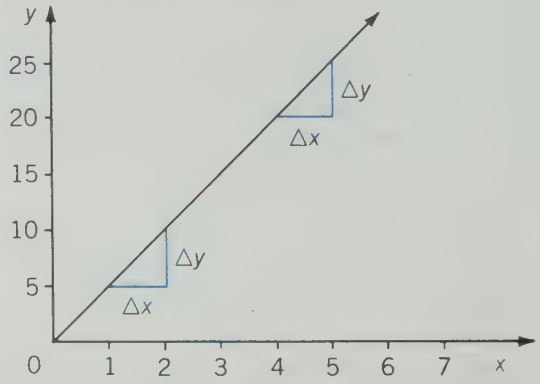


Figure 1-8

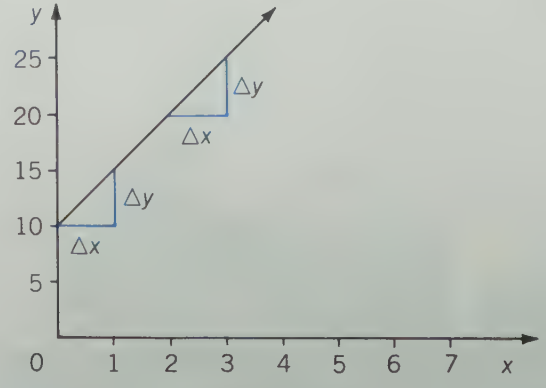


Figure 1-9

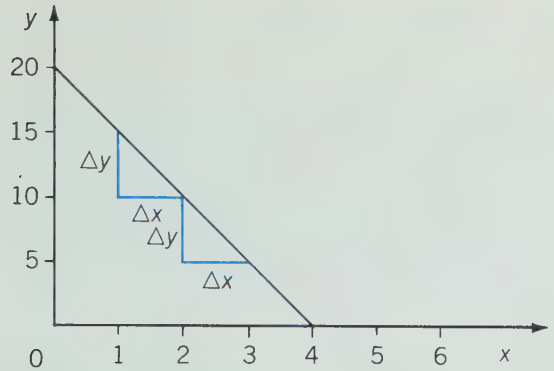


Figure 1-10

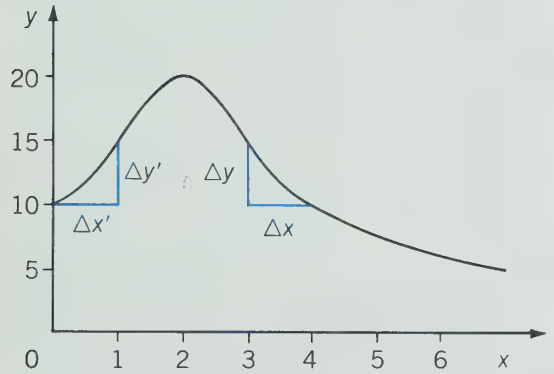


Figure 1-11

In Figure 1-8 what is the y-value when $x = 2$? when $x = 4$? when $x = 6$? When x doubles how does y change? When x triples how does y change? If x increased 5-fold what change would there be in the y-value? If x increased n-fold what would be the change in y? This is an illustration of a direct variation.

You will have noticed that the point (0, 0) is contained in the graph of a direct variation.

> A graph of a direct variation goes through the origin and is linear.

In Figure 1-9, does the value of y double when the value of x doubles? Does the value of y triple when the value of x triples? Is this the graph of a direct variation? When the x-value increases from 1 to 2 (increases by 1) how much does the y-value increase? Does a unit increase in x always produce the same change in y? Test several cases. How much does the y-value change when x increases from 1 to 3? from 1 to 4? by 2? by 3? by n? This is an illustration of a partial variation.

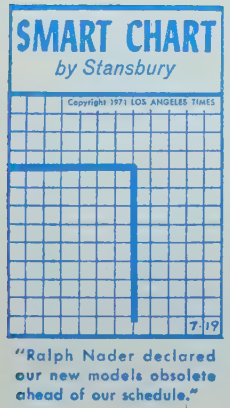
> The graph of a partial variation intersects the y-axis at some point other than zero. The graph of a partial variation is linear.

The slope of a graph showing direct, or partial variation is constant. If Δy represents the change in y and Δx represents the change in x then:

$$\frac{\Delta y}{\Delta x} \text{ is constant.}$$

Notice, Figure 1-11, $\frac{\Delta y'}{\Delta x'}$ is positive while $\frac{\Delta y}{\Delta x}$ is negative. The slope changed.

Pertinent facts about the four graphs shown are listed in the chart below. Study the chart carefully and be prepared to support the comment in each square.

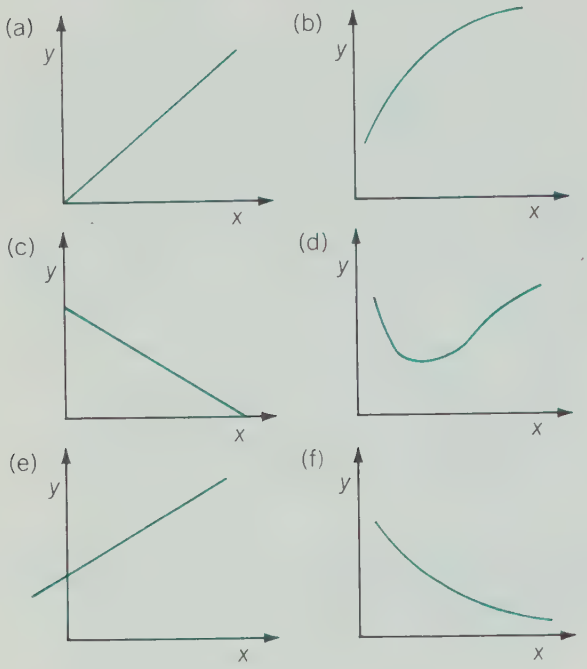


"Ralph Nader declared our new models obsolete ahead of our schedule."

	FIGURE			
	1-8	1-9	1-10	1-11
If change in x is 1, then change in y is ■.	5	5	−5	changes
Slope $\left(\frac{\Delta y}{\Delta x}\right)$	constant (5)	constant (5)	constant (−5)	changes
Type of graph	linear	linear	linear	nonlinear
Type of variation	direct	partial	partial	
Type of relation	function	function	function	function
y-intercept	0	10	20	10

EXERCISE 1·4

1. Tell as much as you can about the type of relation and variation in each of the following.



2. Refer to the situation in the problem of question 3 of Exercise 1·3. Draw a table to show the number of feet (F) between the level of the water and the top of the tank if there is 3 feet of water in the tank when the water starts to flow.
   (a) Draw the graph of this relation.
   (b) What is F after 3.5 minutes? after 20 minutes?
   (c) How long does it take to fill the tank?
   (d) What is the amount of air in the tank after $3\frac{1}{2}$ minutes? after 20 minutes?
   (e) Determine the change in F for each change of 1 in t. Is this change in F constant?
   (f) Does F double when t doubles? Does F become half as large when t doubles?
   (g) Is the graph linear?
   (h) Is the relation an example of a direct or partial variation? Explain.

3. Marty, on his motorbike, made a *circle* tour by riding 40 miles east, then 40 miles north, then 40 miles west and finally 40 miles south. He travels 40 miles per hour.

   Complete the table.

TIME IN HOURS (t)	0	1	2	3	4
SHORTEST DISTANCE IN MILES FROM HOME VIA THE ROAD (D)					

   (a) Draw the graph of D with respect to t (t is on the x-axis). Will you draw a continuous graph? Why?
   (b) Is D a linear function of t? Why?
   (c) Draw the graph of t with respect to D. (D is represented on the x-axis.)
   (d) Is t a linear function of D? Why?
   (e) Is this an example of direct variation? of partial variation?

4. In the above question, let D^1 be the distance Marty has travelled. Make a table and draw the graph of the relation of D^1 with respect to t for $0 \leq t \leq 4$. Is this an example of a direct variation? of a partial variation? Support your answer.

1·5 A Space Lab

Project

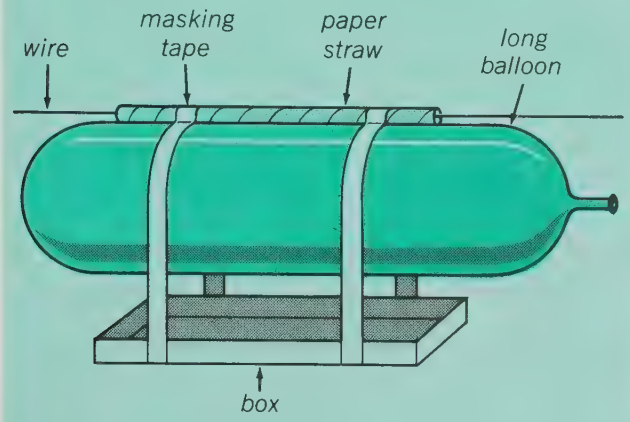
Project 2

Purpose

To determine the relationship between the *load* and the distance a jet-propelled vehicle travels on a given amount of *fuel.*

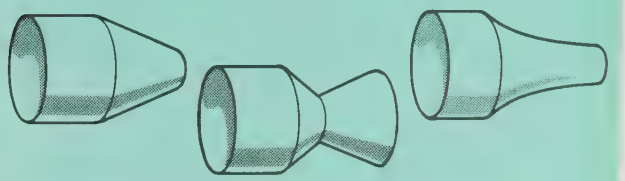
Method

This is left to the student. The diagram gives a hint.

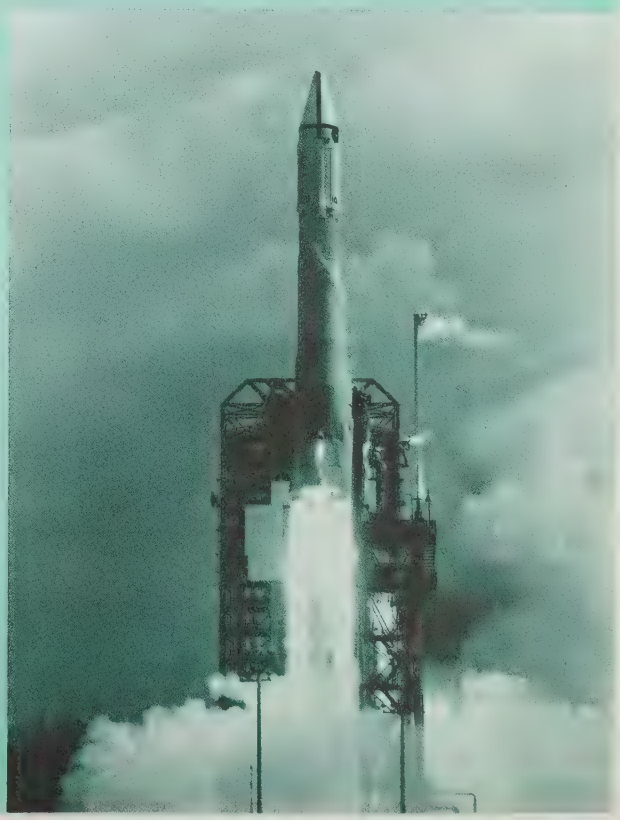


Questions

1. How will you control the *fixed* amount of *fuel?*
2. How will you alter the load?
3. How will you determine the distance?
4. How many ways can you present the data?
5. Which way tells the *story* the best?
6. Is the relation between *load* and *distance* direct or inverse? linear or otherwise?
7. Can you develop an algebraic formula for the relation?
8. Do you think your experiment is affected by other variables than just *load* and *distance*? If so, what?
9. What effect, if any, is there on the relation when the wire is inclined at various angles?
10. Do you think the shape of the *nose cone* affects the distance? Devise an experiment to test the effect of the shape of the nose cone. Keep weight constant.
11. Devise an experiment to test the effect of the shape of the exhaust nozzle. (Keep weight constant.)



12. Can you think of other ways of relating distance travelled to load carried with a fixed *fuel* load? (Electric fan connected to a solar cell mounted on a roller skate—expose to a strong light in a darkened room for a given period of time; slingshot, with fixed pull.)



EXERCISE 1·5

Specific Impulse of a propellant for rockets is a measure of the amount of thrust per unit weight of propellant for unit time of burn.

$$\text{Specific Impulse} = \frac{\text{thrust in pounds} \times \text{length of burn in seconds}}{\text{weight of propellant in pounds}}$$

Specified Impulse has the same general meaning to a rocket engineer as *miles per gallon* has to the driver of an automobile.

Example

What is the *S.I.* of 200,000 pounds of propellant that produces a thrust of 250,000 pounds in burning for 80 seconds?

Solution

$$S.I. = \frac{\text{thrust} \times \text{time}}{\text{weight}}$$

$$= \frac{250{,}000 \times 80}{200{,}000}$$

$$\therefore S.I. = 100 \text{ seconds}$$

1. Find the Specific Impulse of 100 tons of fuel which produces a thrust of 500,000 pounds in 50 seconds of burn.

2. The Saturn V launch vehicle used a liquid propellant which produced a thrust of 7,500,000 pounds. If the weight of the propellant was 35 tons and the Specific Impulse was 200 seconds, how long would the burn be?

3. As the thrust increases, what do you expect happens to the Specific Impulse?

4. As the weight increases, what do you expect happens to the Specific Impulse?

5. Apollo 11 weighed 3100 tons. It had 7.6 million pounds of thrust. The time of the burn was 2 minutes, 40 seconds. There were 559,218 gallons of liquid oxygen and kerosene for propellant. The average weight of a gallon of propellant was 16 pounds. Find the Specific Impulse.

1·6 Interpolation and Extrapolation

A manufacturer of men's shoes wishes to establish a relation between the length and width of the foot of average people. Let us try in the next two sections to establish such a relation.

Project

Project 3

Purpose

To establish a relation between the length and width of an average foot.

Method

1. Working in groups, choose any 10 boys in the class. (Preferable to choose so that boys with small feet as well as boys with large feet are picked.)
2. Trace on a sheet of paper the outline of the right foot of each person. (One tracing of each foot for each group.) Trace the foot without the shoe on. Keep the pencil vertical when tracing the outline.
3. Record in a table of values the length and width of each foot outline (accurate to 0.1 cm).
4. Record the data on a graph with the conventional x-axis the length l.
5. Determine whether or not the information you have obtained constitutes a function. (If it is, you may not be able to identify the function.)

Retain the information and graphs you have obtained. We will come back to them later. In the meantime, follow through the next sections.

When you gathered the information in the above project, you possibly could not see much of a relation except that, generally, as the length of the foot increased, the width also increased. (What kind of a relation does this indicate?) Following is the data collected by one group of 7 students:

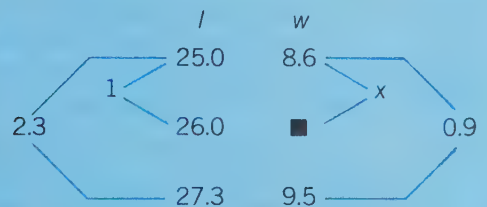
Table 1

l	25.0	27.3	28.5	28.5	29.5	32.0	33.1
w	8.6	9.5	9.9	10.2	10.9	12.1	13.1

Can we tell from Table 1 what the approximate width of a foot is when the length is 26.0 cm? The measure 26.0 lies between 25.0 and 27.3. From the chart, then, it is likely the width is between 8.6 and 9.5.

l	w
25.0	8.6
26.0	■
27.3	9.5

Let us set up a proportion using the differences shown.



Let the difference in width as shown be x cm. That is, the width will increase by x cm when the length increases by 1 cm. The terms of the proportion are:

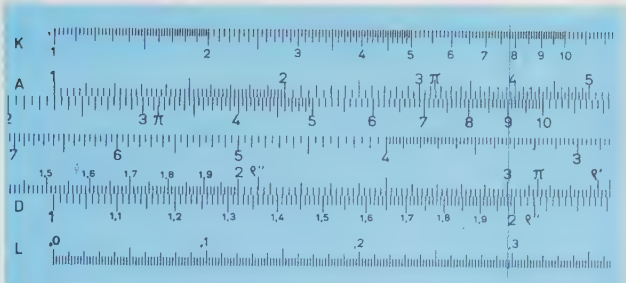
First term *Second term*

$$\frac{1 \text{ (increase in length)}}{2.3 \text{ (total increase)}} = \frac{x \text{ (increase in width)}}{0.9 \text{ (total increase)}}$$

Solving: $$\frac{1}{2.3} = \frac{x}{0.9}$$

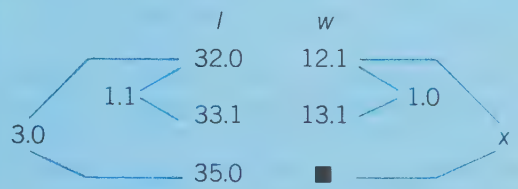
On your slide rule, set the 1 on the *A* scale against the 2.3 on the *B* scale, then find 0.9 on the *B* scale. The corresponding value on the *A* scale is x. The value on the *A* scale, you will find, is .39. Therefore x is 0.39.

A scale → $\frac{1}{2.3} = \frac{x}{.9}$ ← *A* scale
B scale → *B* scale



Recalling that we are working with measures accurate to the nearest tenth of a centimetre, we round 0.39 to 0.4. The foot that is 26.0 cm long is 8.6 + 0.4 or 9.0 cm wide.

Notice that we now have more data than before. We found data in the range between two given sets of data. We can find data beyond that given in the table in a similar manner. Let us determine the width of a foot that is 35.0 cm long. (Does anyone have a foot that long?)



The terms of the proportion are:

$$\frac{1.1 \text{ (increase in length)}}{3.0 \text{ (total increase)}} = \frac{1.0 \text{ (increase in width)}}{x \text{ (total increase)}}$$

Use the slide rule to solve:

$$\frac{1.1}{3.0} = \frac{1.0}{x}$$
$$x = 2.7.$$

Thus, the width of the foot is 12.1 + 2.7 or 14.8 cm. (Do you think this is reasonable?)

The process of finding new information located between the measured data is called *interpolation*. The process of finding new information *beyond* the measured data is called *extrapolation*. Interpolation and extrapolation are based on the assumption that the pattern established by the individual pieces of data will hold when extended to cases between or beyond those cited. There is always a danger in interpolating and that risk or danger is increased when extrapolating. This will be borne out in some of the questions in later exercises.

EXERCISE 1·6

Use the values in Table 1.

1. Interpolate to find the width when the length is:
   (a) 28.0 cm (b) 29.0 cm (c) 31.0 cm.

2. Extrapolate to find the width when the length is:
   (a) 34.0 cm (b) 34.5 cm
   (c) 36.0 cm (d) 24.0 cm.

3. Interpolate or extrapolate to find the length when the width is:
   (a) 11.0 cm (b) 10.5 cm
   (c) 7.5 cm (d) 13.5 cm.

4. When you drew your graph did you use a bar graph, a dot graph, a continuous line graph, or some other graph? Explain why you chose the one you did. Do you think it was the best choice? Why?

5. Use a dictionary to look up the words *interpolate* and *extrapolate.* Study both the mathematical and nonmathematical definitions. How are the nonmathematical meanings related to the mathematical meanings?

1·7 The Curve of Best Fit

We still have not defined the relation between the length and width of the average foot. Let us pursue the topic further.

The data collected by the group of seven is recorded in this graph in Figure 1-12.

Figure 1-12

If we place a clear plastic straightedge along the line of dots, the dots appear to almost fit a straight line. The line does not contain many of the points but it is not far from any point either. Notice that line *a* is a *better fit* than line *b*.

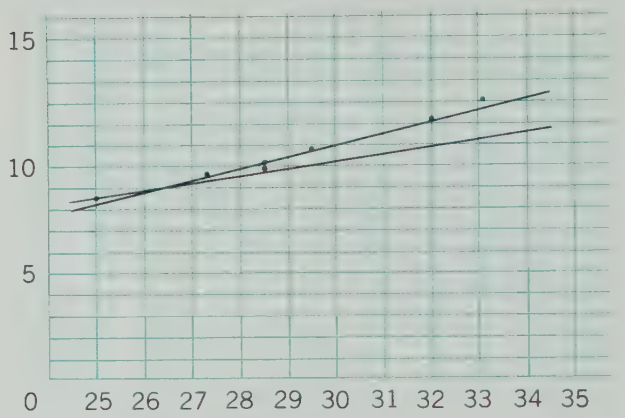


Figure 1-13

When we say line *a* is a better fit than line *b*, we mean that line *a* is closer to all the points, on the average, than is *b*. For instance, we will call the distance a point is above the line, positive, and the distance it is below the line, negative. The set of points are then + .3, − .1, − .2, + .1, 0, − .1, − .1, from line *a* or an average of $\frac{9}{70}$, whereas, they are 0, 0, 0, + .3, + 1.0, + 1.5, + 2.0, from line *b* or an average of $\frac{48}{70}$.

We can now use line *a* to interpolate to find the width of a foot that is 26.0 cm long. We move vertically from the 26 cm mark on the horizontal axis to line *a*. From this point, we move horizontally to the vertical axis. We read the value of the width as 8.8 cm. Notice this interpolated value is less than that calculated earlier. Similarly, we can extrapolate to find the width when the foot is 35.0 cm long. The width is found to be 13.9 cm.

Now that we have a linear graph that expresses the relation, we will be able to write an algebraic expression for the relation. You will recall from earlier work that if the *y*-intercept and the slope of a line are known, the equation can be written in the form $y = mx + b$ where m is the slope and b is the *y*-intercept. In the situation above, the *w*-intercept is − 7 approximately. The slope is found by using a triangle as shown in Figure 1-14.

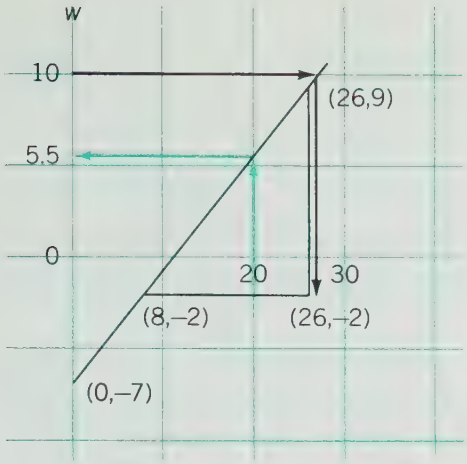


Figure 1-14

$$\text{Slope} = \frac{\triangle w}{\triangle l}$$

From the graph in Figure 1-14, change in w is 11, change in l is 18. The slope is $\frac{11}{18}$. The algebraic function then, is

$$w = \tfrac{11}{18} l - 7.$$

It may be written

$$F = \{(l,w) | w = \tfrac{11}{18} l - 7, l \in R\}.$$

The notation $F(20)$ means the value of the function (w) when l is 20.

Example 1

Find (a) $F(20)$ (b) $F(30)$.

Solution

(a) Using the graph we find $F(20) \simeq 5.5$ (See colored line on graph.)

(b) $F(30) \simeq 11.5$

Example 2

Find l when (a) $F(l) = 10$ (b) $F(l) = 16$.

Solution

(a) Using the graph we find if $F(l) = 10$, then $l \simeq 28$. (See dotted line on graph.)

(b) If $F(l) = 16$, then $l \simeq 37$.

EXERCISE 1·7

1. Use the graph in Figure 1-13 (line *a*) to interpolate and extrapolate for the values of questions 1, 2 and 3 of the previous exercise. Can you account for the difference?

2. Use your graph in Project 3. Draw a *best fit* curve for the data.
   Find
   (a) $F(25.0)$ (b) $F(27.5)$
   (c) $F(32.4)$ (d) $F(35.0)$

3. Use the best fit curve in question 2 to find l in each case:
   (a) $F(l) = 5.5$ (b) $F(l) = 6.0$
   (c) $F(l) = 7.7$ (d) $F(l) = 15.0$.

4. Repeat Project 3, but choose 10 girls from the class. Then repeat questions 2 and 3 above.

5. What do you notice about the *average* foot of a boy compared to the *average* foot of a girl.

6. Use the best fit curve in questions 2 and 4 to find the y-intercept and slope in each case.

Cartoon courtesy Better Homes and Gardens magazine. © Meredith Corporation, 1967

"I undercooked it so it won't be overcooked when I warm it over tomorrow."

Develop the habit of valid reasoning. The ability to reason will help you in most fields of study.

7. Use the data given to do the following exercise:

x	2.1	4.0	5.0	7.0	9.0
y	10.0	15.1	22.0	26.9	30.1

(a) Record the information on a graph appropriate for answering the rest of these questions.

(b) Find y, by means of proportions, when x is (i) 3.2 (ii) 8 (iii) 0 (iv) 14.

(c) Draw the best fit curve on the graph in (a).

(d) Repeat (b) using the best fit curve instead of proportions.

(e) Use the best fit curve to find the function relating x and y.

(f) Why do your answers in (b) and (d) not always agree?

1·8 More Interpolation and Extrapolation

Suppose we measure the distance a ball falls from rest in successive seconds (distance against time). From the measurements we obtain the distance at certain times as shown in Table 1.

Table 1

t (sec)	0	1	2	3	4	5
d (ft)	0	16	64	144	256	400

The graph is shown in Figure 1-15.

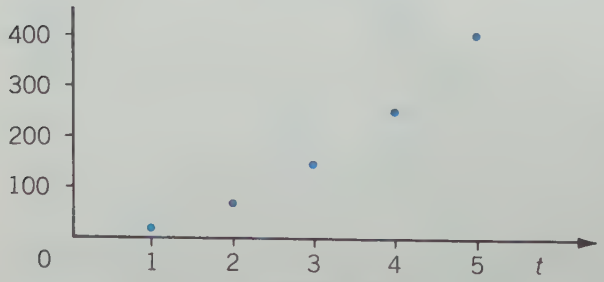


Figure 1-15

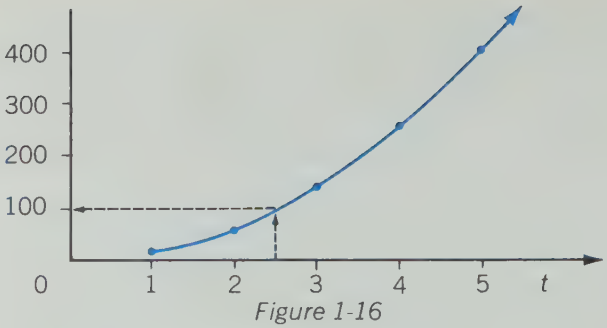


Figure 1-16

If we draw a smooth curve to contain the points, we have a continuous graph. This is the *best fit* curve. We are able to find the distance the ball has fallen for new times between those we have measured. In Figure 1-16, we have shown how to find the distance the ball has fallen in $2\frac{1}{2}$ seconds. It has fallen 100 feet.

Usually experimental data contains errors. These errors may be due to measurement or inadequacies of the apparatus, or other factors. Hence, situations may occur that give results as shown in Figures 1-17 and 1-18.

Galileo making his experiments on falling bodies from the leaning Tower of Pisa. "They were seen to fall evenly."

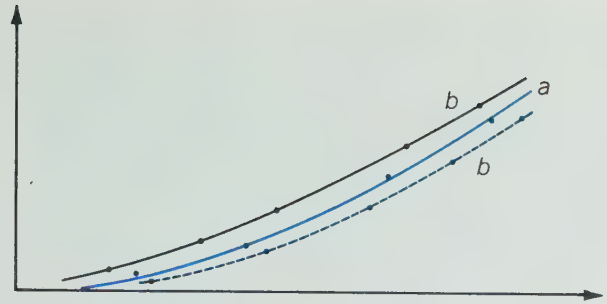


Figure 1-17

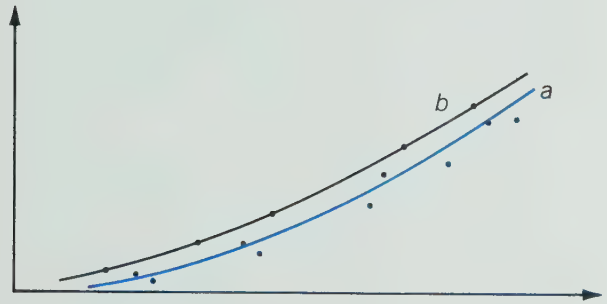


Figure 1-18

Notice that not all points are contained in the best fit curve. Curve *a* fits the data best in that it is not too far from any of the points (on the average). Lines *b* in Figures 1-17 and 1-18 may be closer to some points but the average distance is greater than for lines *a*.

EXERCISE 1·8

1. Use proportions and the table of values in this section to find the distance the ball falls in:
   (a) $3\frac{1}{2}$ seconds (b) $4\frac{3}{4}$ seconds (c) 6 seconds.

2. Use the graph in Figure 1-16 to interpolate or extrapolate to find the distance the ball falls in:
   (a) $3\frac{1}{2}$ seconds (b) $4\frac{3}{4}$ seconds (c) 6 seconds.
   Compare with those in question 1.

3. How long does the ball fall in order for it to fall:
   (a) 50 feet (b) 100 feet
   (c) 150 feet (d) 200 feet?

4. Consider the ball in the preceding section. Consider it to have been dropped from a height of 600 feet (Table 1).
   (a) Set up a table of values to show the relation between s – the distance from the ground, and t – the time it has fallen.
   (b) Plot the points on a graph.
   (c) Draw the best fit curve on the graph in (b).
   (d) Interpolate to find the distance the ball is from the ground at the end of $3\frac{1}{2}$ seconds.
   (e) At what time is the ball 300 feet from the ground?
   (f) Extrapolate to determine when the ball will strike the ground.

5. Refer to question 3, Exercise 1-4. Make a table and draw a graph for the relation between t and D for $0 \leq t \leq 4$ where D is the distance as the crow flies (shortest across-country distance) Marty is from home.
   (a) Did you draw a continuous line graph? Should the lines be straight? To help answer this last question go on to part (b).
   (b) Draw segments between each set of adjacent points on the graph. Interpolate to find D when $t = 1.3, 1.5, 1.7$.
   (c) Draw a diagram to show the relative position of Marty from home when $t = 1.3$. How far east of home is he? How far north of home is he? Use the Pythagorean property to find how far he is from home. Does this agree with the answer you obtained in (b)? Repeat for $t = 1.5$ and $t = 1.7$. Do these answers agree with those in (b)?
   (d) Use the calculator to find D when t is 1.1, 1.2, 1.4, 1.6, 1.8, 1.9. Now use this data with that you already have to draw the graph of D varying with t. Use a smooth curve to join the points $t = 1$ through $t = 3$.
   (e) What kind of a variation exists between t and D for $0 \leq t \leq 1$ and $3 \leq t \leq 4$?
   (f) For $1 \leq t \leq 3$, is this relation above a direct variation? a partial variation? Why?

6. Care must be exercised when extrapolating. The following data gives the volume of a certain mass of gas as the temperature of the gas varies. Draw the graph from the data.

T	10°	40°	60°	85°	95°
V	26.2	29.1	31.2	33.7	35.0

Arrange the axes so that temperatures between − 300 and + 100 are plotted along the x-axis and volumes between 0 and 35 are shown on the y-axis.

(a) By extending the graph and extrapolating, determine the volume when the temperature is − 275°. Examine your answer. Is this a reasonable explanation of the physical situation? Can you explain what really happens? If not, ask your science teacher.

(b) Is this a linear relation? Why?

7. Airplanes and spacecraft flying in air at high speeds create heat on the surface.

At 700 mph (speed of sound),
increase of temperature ≃ 95°F.
At 10,000 mph,
increase of temperature ≃ 1000°F.
At 17,000 mph,
increase of temperature ≃ 3000°F.
At 25,000 mph,
increase of temperature ≃ 8500°F.

(a) Estimate temperature at
(i) 5000 mph
(ii) 5 x speed of sound (mach 5)
(iii) 30,000 mph.

(b) What speed is necessary to create a temperature of 200°F?

(c) Aluminum vaporizes at 2450°F. What speed would cause aluminum on the wings to vaporize (if no protection were offered)?

8. The formula $P = \frac{1}{2}A + 110$ gives the normal blood pressure (P) for a person A years old.

(a) Make a table of values relating P and A where A is in multiples of 10, $0 \leq A \leq 70$.

(b) Construct a graph using the data in (a). Should this be a continuous line graph? Why?

(c) From the graph determine the normal blood pressure for a person

(i) 15 years old
(ii) 25 years old
(iii) 95 years old.

(d) Is P a linear function of A? Why?

(e) What kind of variation is this?

(f) Obtain a sphygmomanometer (instrument to record blood pressure) from the biology laboratory. Check your blood pressure. Does it agree with the information in this question? Explain.

9. The pressure of wind against a vertical surface increases as the wind increases. The actual data is recorded in the table.

Wind velocity in mph (V)	20	40	60	80	100	120
Pressure in lbs per sq ft (P)	2	8	18	32	50	62

(a) Draw a suitable graph using the above data.

(b) Use the graph to determine the pressure when the wind velocity is
(i) 30 mph (ii) 50 mph (iii) 90 mph.

(c) Use the graph to determine the pressure when the wind velocity is
(i) 10 mph (ii) 150 mph.

(d) Check your answers from (b) and (c) by using the formula $P = .005V^2$.

Project

Project 4

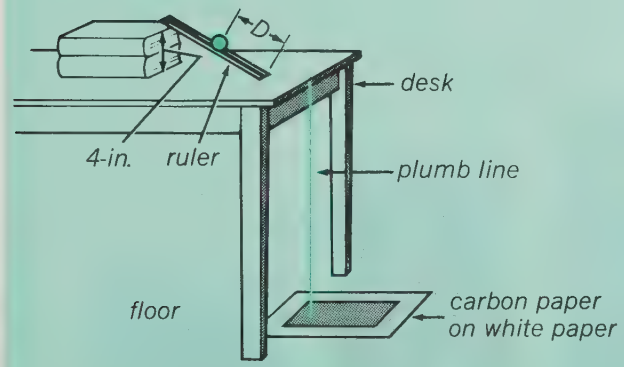
Purpose

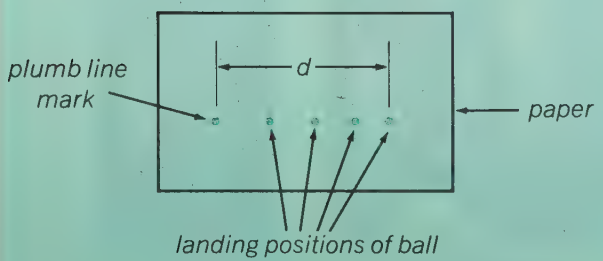
To graph the relation between the distance, D, a ball moves down an incline plane and the distance from the table edge, d, the ball falls, and the effect the slope of the incline plane has on the distance d.

Part A

Method

Set up a 1-foot ruler with a groove down the middle as shown. The upper end (12-inch mark) is 4 inches above the table top. The lower end is 1 inch from the edge of the desk. Let a steel ball (or large marble) roll down the ruler from the 1-inch point. Let it hit the paper on the floor.





(a) Record the distance the ball landed from the plumb line mark. Set up a table of values for original ball position D, 1 inch through 12 inches. Record the distance d the ball landed from the plumb line mark.

(b) Plot the points on a graph.

(c) Draw the *best fit* curve. (The best fit curve does not have to contain all the points in the data.)

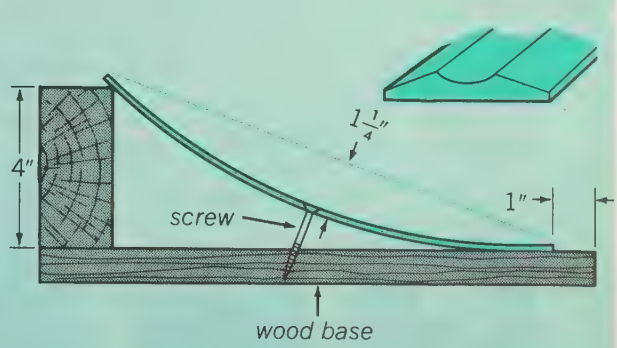
(d) Extrapolate to find the distance d when D is
(i) 15 inches
(ii) 24 inches
(iii) 36 inches.

(e) Interpolate to find the distance d when D is
(i) $3\frac{1}{2}$ inches
(ii) $5\frac{1}{2}$ inches
(iii) $9\frac{1}{2}$ inches.

(f) Do you think the extrapolation of $D = 36$ is very accurate? You might test your extrapolated figure by replacing the 1-foot ruler by two yardsticks forming a V. Be careful to keep the slope constant.

(g) Is this a graph of a linear function? Why?

Part B



Method

For an interesting variation of this project, use a plastic ruler and set it up as shown. The plastic ruler usually has a hole in the middle. Use a small screw with a flat head so that the ball does not hit it. There should be about $1\frac{1}{4}$" bow in the ruler. Obtain data to plot d against D.

Draw a dot graph. Draw the best fit curve.

(a) Answer parts (d) and (e) in Part A.

(b) Is the graph that of a linear function? Why?

1·9 Caution! Interpret Graphs Carefully!

Everywhere one looks today he sees graphs that are designed to carry a message to the public. Sometimes the graphs mislead or leave an incorrect impression. The graphs in these exercises have been taken from actual cases and modified to meet the needs of this text. The *misleading* features occur as they did in the actual advertisement.

EXERCISE 1·9

1. Identify the misleading features of this magazine advertisement.

Answer the following to help you identify the misleading features.

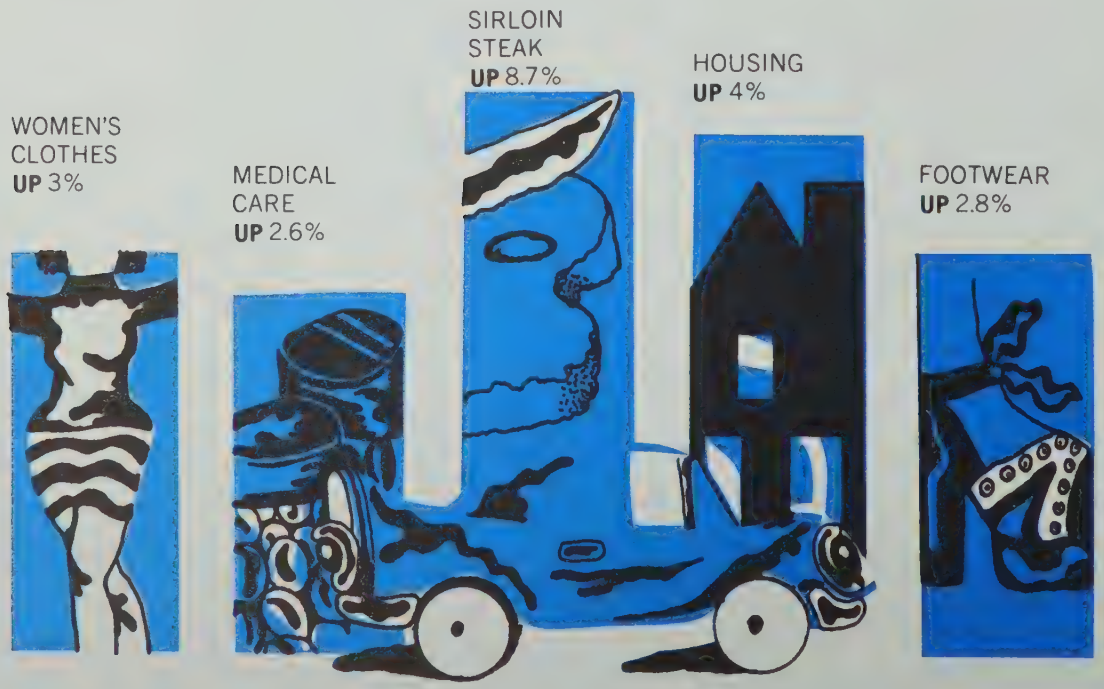
(a) From looking at the heights of the columns, has the cost of medical care gone up about the same as the cost of footwear? Does it appear that sirloin steak went up less than twice what footwear did? Should it?

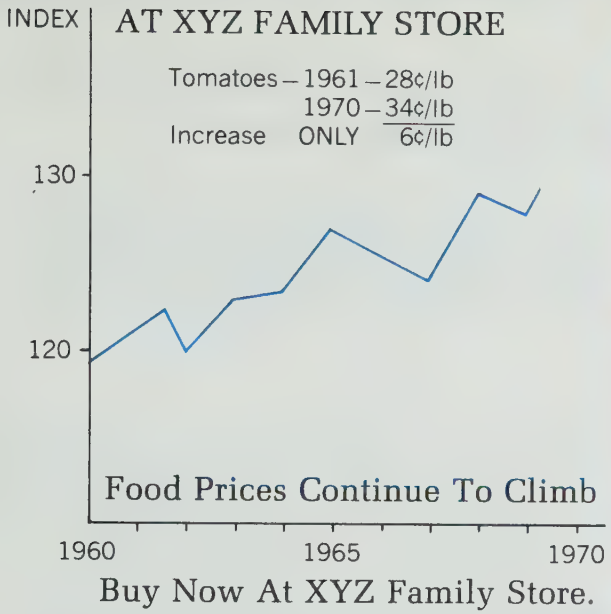
(b) Now look at the actual percentages the medical care, footwear and steak have gone up. Are the heights proportional to the percentages?

(c) What inference does the advertiser want you to make?

(d) Is the *message* that the public *reads* from the advertisement supported by the data or does the advertisement ask the public to jump to a false conclusion?

(e) What conclusion does the advertisement want the public to make?





2. Identify the misleading feature of the above advertisement.

   (a) Compare the height of the index in 1961 with that in 1970. (Do not refer to the numbers on the left but just compare the distance of each from the base line.) Does it appear that the *prices* doubled?

   (b) Now refer to the index numbers (which really do not mean the same as prices) on the left. Compare the index in 1961 with the index in 1970. Did the index double? What percent increase is the index in 1970 over the index in 1961?

   (c) What message do you think Family Store XYZ wanted *John Q. Public* to infer?

   (d) What percent did tomato prices increase from 1961 to 1970?

   (e) Do you think the facts and the *message* contradict each other? Explain.

3. Study local magazines, newspapers and advertising to obtain examples in which graphs are used to mislead the public in the two ways studied in this chapter or in other ways. Be prepared to explain wherein the advertisement *misleads*.

Using the Library

1. *How To Lie With Statistics* (Huff)
   Read pages 60-73. Describe or illustrate how graphical representations of information can be misleading.

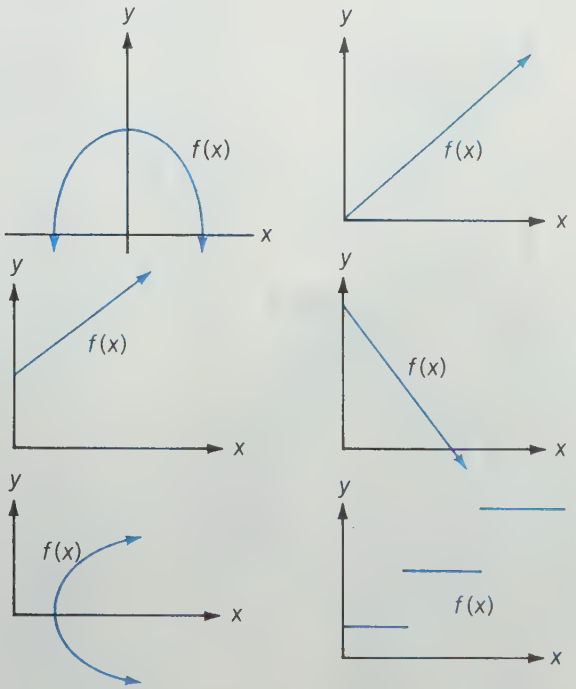
2. *How To Lie With Statistics* (Huff)
   Read pages 11-26. Can you present some statistics collected in your class which although based on actual data may be misleading because of the sample?

Examples

The average mark in math in your class.
The average income in your class.
The height of the boys in your class.

REVIEW EXERCISE 1

1. Identify each relation by using the terms direct variation, partial variation, function, nonfunction, nonlinear, linear, etc.

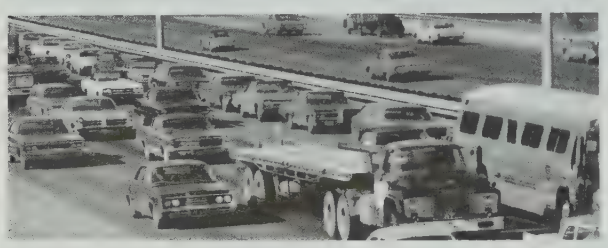


2. Given

x	2	6	10	14
y	8	18	28	38

(a) Use proportions to find x when
(i) $y = 15$ (ii) $y = 42$.
(b) Graph the information.
(c) Describe the relation.

3. It is recommended that a driver leave one car length (17 feet) between his car and the car ahead for each 10 mph of speed. Show distance between cars for various speeds by:
(a) table of values
(b) the most appropriate graphic representation.
Tell why you used the method you did in (b).



To The Student

A number of business firms were asked for a description and sample of the types of problems they would expect employees to be able to do.

The following is a sample of an actual order filled by a lumber company. A board foot is a volume of lumber measuring 1 ft x 1 ft x 1 in. For example, the piece of spruce in the first line is 2" x 4" and 12' long. Therefore its board measure is

$$2'' \times \tfrac{4}{12}' \times 12' = 8 \text{ board feet}$$

and the total quantity is

$$14 \times 8 = 112 \text{ board feet.}$$

Abbreviations used are:
MBM: per thousand board measure,
M sq ft: per thousand square feet,
Lin ft: per lineal foot.
Thus the first item at $140.00 MBM would cost

$$\frac{112}{1000} \times \$140 = \$15.68.$$

Complete the AMOUNT column and calculate the total bill for the lumber.

Ed's LUMBER

QUOTATION

№ 21

This estimate is given for acceptance within 15 days of date. Any material or labor not specifically listed will be supplied at addition cost to purchaser.

ACCEPTANCE:
The above offer is accepted on the terms specified

..... DATE ..... 19.....

..... Purchaser.

We agree to furnish the material listed for the sum of $ .....

($ .....) Terms .....

USE		DESCRIPTION	QUANTITY	PRICE	UNIT	AMOUNT	
FLOOR							
FRAMING	14	PCS 2X4 12'-0 SPRUCE	112	$140.00	MBM		
	36	LIN 2X2	12	142.00	"		
PLYWOOD	7	SHTS. 4X8 · 5/8" FIR	224	252.00	SQ FT		
WALLS							
PLATES	120	LIN FT. 2X4 SPRUCE	80	140.00	MBM		
	60	PCS 2X4 - 8'-0 "	320	140.00	"		
CEILING	180	LIN FT 1X3 SPRUCE	45	135.00	"		
	216	SQ FT 12X12 - 1/2"	216	16.4	SQ FT		
		CEILING TILE					
FINISH MATERIAL	1	SET 1X5 DOOR JAMB	1	3.35	EA		
	1	SET 3/8 X 1 1/4 DOOR STOP	1	85	EA		
	1	ONLY 2 3/8 X 6 8 1 3/8	1	4.50	EA		
		FIR PAINT GRADE DOOR					
	60	LIN FT 2 1/4 BASE BOARD	60	.04	LIN FT		
	60	LIN FT 3/4 X 3/4 FIR COVE	60	.03 1/2	"		

Chapter Two
EQUATION OF A STRAIGHT LINE

2·1 The Straight Line

In this chapter the straight line is studied by comparing various forms of linear equations and their respective graphs. The universe for the variables is the set of real numbers unless indicated otherwise.

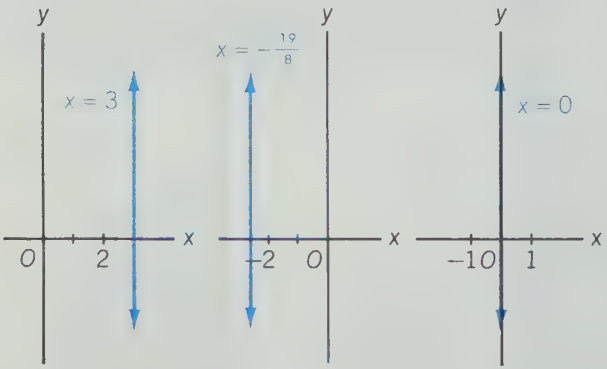
Review

Linear graphs may be grouped into three sets. The general forms of the three types are:

Case I $\{ (x,y) \mid x = a \}$ (vertical lines)
Case II $\{ (x,y) \mid y = b \}$ (horizontal lines)
Case III $\{ (x,y) \mid y = mx + b \}$ (sloping lines)

Case I. The three graphs in Figure 2-1 along with their respective equations illustrate the graph of

$x = a$.



(*i*) (*ii*) (*iii*)

Figure 2-1

Note:

(a) In the general form $x = a$, the a indicates the x-intercept.

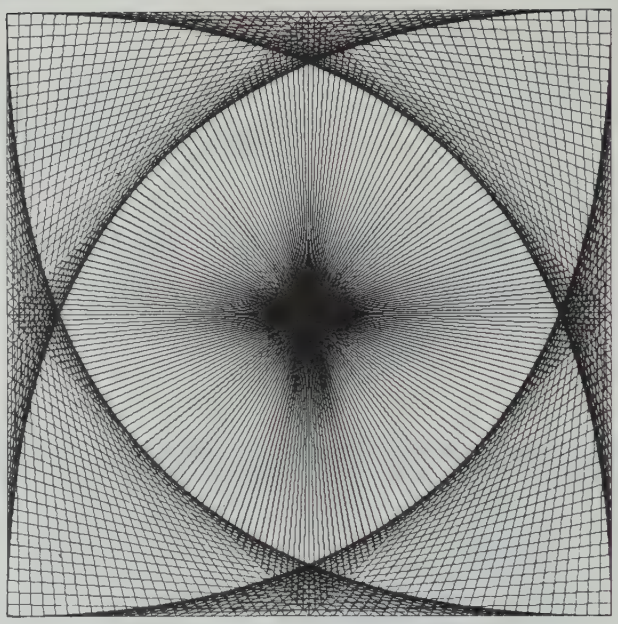
(b) The line is vertical: perpendicular to the x-axis.

(c) Domain: (i) $\{3\}$ (ii) $\{-\frac{19}{8}\}$ (iii) $\{0\}$

(d) Range: R.

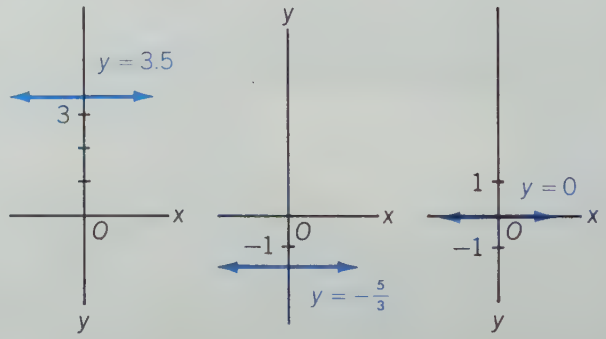
(e) The relations are not functions.

(f) Slope is undefined.



A computer drawing consisting of straight lines. Calcomp test pattern. (California Computer Products, Inc.)

Case II. The three graphs in Figure 2-2 illustrate the graph of $y = b$.



(*i*) (*ii*) (*iii*)

Figure 2-2

Note:

(a) In the general form $y = b$, the b indicates the y-intercept.

(b) The line is horizontal: parallel to the x-axis. Slope = 0.

(c) Domain: R

(d) Range: (i) $\{3.5\}$ (ii) $\{-\frac{5}{3}\}$ (iii) $\{0\}$.

(e) Each relation is a function.

Case III. The three types in Figure 2-3 illustrate the graph of $y = mx + b$.

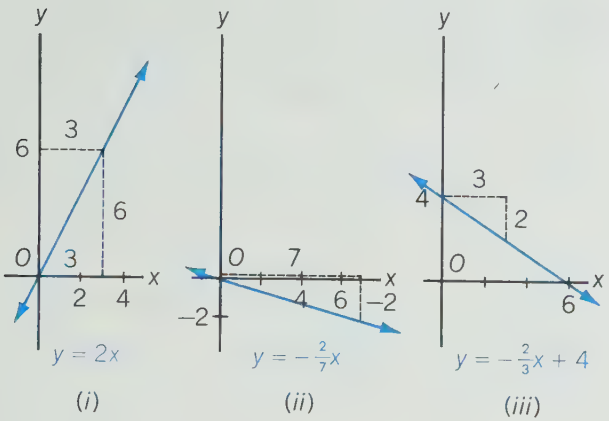


Figure 2-3

The standard form for equation (i) is $y = 2x + 0$ and for equation (ii) is $y = -\frac{2}{7}x + 0$. Recall that slope is $\frac{\triangle y}{\triangle x}$.

Note:

(a) In the general form, b indicates the y-intercept. The y-intercept in each is:
(i) 0 (ii) 0 (iii) 4.

(b) In the general form, m indicates the slope. The slope of each is given and a check is shown.

(i) m = slope = 2 $\quad \frac{\triangle y}{\triangle x} = \frac{6}{3} = 2$

(ii) m = slope = $-\frac{2}{7}$ $\quad \frac{\triangle y}{\triangle x} = \frac{-2}{7} = -\frac{2}{7}$

(iii) m = slope = $-\frac{2}{3}$ $\quad \frac{\triangle y}{\triangle x} = \frac{-2}{3} = -\frac{2}{3}$

(c) Domain: R.

(d) Range: R.

(e) Each relation is a function.

Equations of this third type do not always occur in the standard form. You will find it best to write any equation in standard form.

Example 1

Write each in standard form.

(a) $x - 3y = 7$ (b) $2y - 10 + x = 0$

Solution

(a) Remember, we want to write this equation in the form $y = mx + b$.

$$x - 3y = 7$$

To transpose the x term to the right side, add $-x$ to each side.

$$x - x - 3y = -x + 7$$
$$-3y = -x + 7$$

To obtain y divide each side by -3

$$\frac{-3}{-3}y = \frac{-x+7}{-3}$$

$$y = \tfrac{1}{3}x - \tfrac{7}{3} \quad (m = \tfrac{1}{3},\ b = -\tfrac{7}{3})$$

(b)
$$2y - 10 + x = 0$$
$$2y - 10 + 10 + x - x = 0 - x + 10 \quad \text{(Why?)}$$
$$2y = -x + 10$$
$$\frac{2y}{2} = \frac{-x+10}{2} \quad \text{(Why?)}$$
$$y = -\tfrac{1}{2}x + 5 \quad (m = -\tfrac{1}{2},\ b = 5)$$

EXERCISE 2·1

1. Is the relation $\{(x, y) \mid x = 5\}$ a function? Explain.

2. Is the relation $\{(x, y) \mid y = 2.5\}$ a function? Explain.

3. Draw these three graphs with respect to the same set of axes.
(a) $\{(x, y) \mid y = \frac{3}{2}x + 4\}$ (b) $\{(x, y) \mid y = \frac{3}{2}x + 2\}$
(c) $\{(x, y) \mid y = \frac{3}{2}x - 3\}$
State in words what you observe from the three graphs.

4. Draw these three graphs with respect to the same set of axes.
(a) $\{(x, y) \mid y = \frac{1}{2}x + 4\}$ (b) $\{(x, y) \mid y = 2x + 4\}$
(c) $\{(x, y) \mid y = \frac{7}{2}x + 4\}$
State in words what you observe from the three graphs.

5. Draw the graph of each.

(a) $\{(x, y) \mid y = 4\}$ (b) $\{(x, y) \mid y = -\frac{3}{2}\}$

(c) $\{(x, y) \mid x = -\frac{3}{2}\}$ (d) $\{(x, y) \mid x = \frac{5}{4}\}$

(e) $\{(x, y) \mid y = 3x\}$ (f) $\{(x, y) \mid y = -2x\}$

(g) $\{(x, y) \mid y = -\frac{5}{4}x\}$ (h) $\{(x, y) \mid y = \frac{7}{4}x\}$

(i) $\{(x, y) \mid y = 2x - 2\}$

(j) $\{(x, y) \mid y = -\frac{2}{3}x + 3\}$

(k) $\{(x, y) \mid y = \frac{3}{2}x - 4\}$

(l) $\{(x, y) \mid y = -\frac{5}{6}x + 5\}$

6. Write each of the following in the standard form $y = mx + b$.

(a) $y + x = 8$ (b) $y - 5 = x$

(c) $y + \frac{2}{3}x = 6$ (d) $y + 3 = -\frac{2}{3}x$

(e) $2y + 4 = x$ (f) $\frac{y}{3} - 4 = x$

(g) $8 + 5y = 2x$ (h) $3 - 7y = 14x$

(i) $x - 2y = 4$ (j) $4y = 3x$

(k) $\frac{4}{3}x - 4y = 8$ (l) $\frac{2}{3}x = \frac{1}{3}y$

(m) $x + y = 0$ (n) $x + 2y = 0$

(o) $\frac{2}{3}x - \frac{3}{4}y = 0$ (p) $\frac{5}{2}x - \frac{5}{2}y = 10$

7. The Celsius-Fahrenheit function is given as
$T = \{(C, F) \mid F = \frac{9}{5}C + 32, -40 \leq C \leq 40, C \in R\}$

(a) What is the domain?

(b) Draw the graph of the function.

(c) What is the range?

(d) From the graph read the temperature on the Fahrenheit scale when the Celsius scale reads 50 degrees.

(e) $T(5)$ means the Fahrenheit temperature when the Celsius temperature is 5, (C is replaced by 5); hence $T(5) = 41$.
Use the graph to find
(i) $T(0)$ (ii) $T(-22)$ (iii) $T(18)$.

(f) Use the graph to find C such that
(i) $T(C) = 0$ (ii) $T(C) = 40$
(iii) $T(C) = -40$.

2·2 $Ax + By + C = 0$

The three types of linear relations discussed in the previous section are frequently combined in the general linear relation of the form

$$L = \{(x, y) \mid Ax + By + C = 0, x, y \in R\}$$

where A, B and C are elements of the real numbers with A and B not both zero.

Let us consider three examples of this linear relation.

Example 1

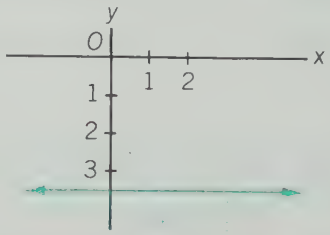
Graph: $L_1 = \{(x, y) \mid 0x + 2y + 7 = 0\}$.

Solution

The equation $0x + 2y + 7 = 0$ can be written in an equivalent form:

$$2y + 7 = 0$$
$$2y = -7$$
$$y = -\frac{7}{2}.$$

We notice this now is of the form $y = b$ as we discussed in the previous section (*Case II*).



In the Arctic the temperature drops as low as –88°F while in the desert it soars to 125°F. State these temperatures in the Celsius scale.

Example 2

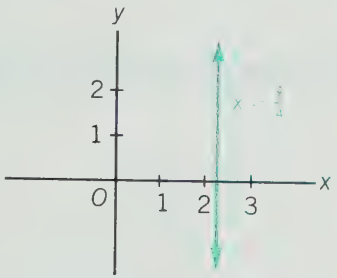
Graph: $L_2 = \{(x, y) \mid 4x + 0y - 9 = 0\}$.

Solution

The equation can be rewritten

$$\begin{aligned} 4x + 0y - 9 &= 0 \\ 4x - 9 &= 0 \\ 4x &= 9 \\ x &= \tfrac{9}{4}. \end{aligned}$$

This is of the form $x = a$ (*Case I* of the previous section)



Example 3

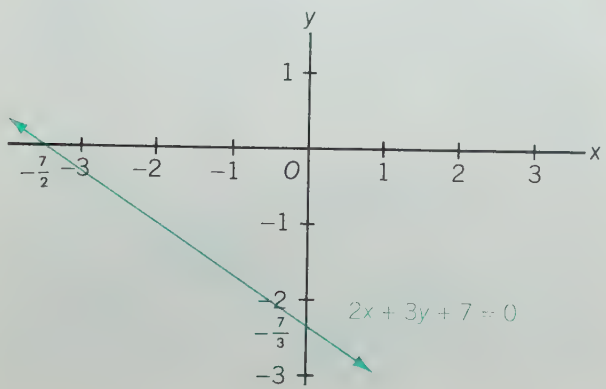
Graph: $L_3 = \{(x, y) \mid 2x + 3y + 7 = 0\}$.

Solution

There are several ways we might do this question. Two methods are shown here.

(a) The *table of values* method.

x	0	$-\frac{7}{2}$	1
y	$-\frac{7}{3}$	0	-3



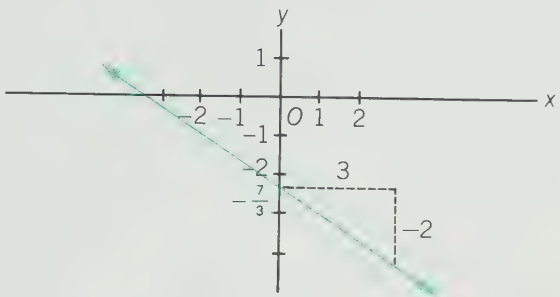
(b) The *y-intercept-slope* method.

First we will write the equation in the form

$$y = mx + b.$$

$$\begin{aligned} 2x + 3y + 7 &= 0 \\ 2x - 2x + 3y + 7 - 7 &= 0 - 2x - 7 \\ 3y &= -2x - 7 \\ \frac{3y}{3} &= \frac{-2x - 7}{3} \\ y &= -\tfrac{2}{3}x - \tfrac{7}{3} \end{aligned}$$

The y-intercept is $-\frac{7}{3}$ and the slope is $-\frac{2}{3}$.



EXERCISE 2·2

1. Write each in the form $y = b$.
   (a) $0x + 3y - 2 = 0$ (b) $0x - 5y + 6 = 0$
   (c) $0x - 4y - 5 = 0$ (d) $0x + By + C = 0$

2. Write each in the form $x = a$.
   (a) $5x + 0y + 2 = 0$ (b) $6x + 0y - 5 = 0$
   (c) $-3x + 0y - 2 = 0$ (d) $Ax + 0y + C = 0$

3. Name the y-intercepts for the graphs in question 1.

4. Name the x-intercepts for the graphs in question 2.

5. Write each in the form $Ax + By + C = 0$.
   (a) $y + 3 = 0$ (b) $x - 8 = 0$
   (c) $x + 2 = 3y$ (d) $y = 2x + 4$
   (e) $y = 3$ (f) $x = -\frac{2}{3}$
   (g) $4y = 3 - x$ (h) $2y = 3x - \frac{2}{3}$

6. Write each in the form $y = mx + b$.

(a) $3x + 2y + 5 = 0$ (b) $2x - 3y - 7 = 0$

(c) $12x - 7y + 12 = 0$ (d) $4x + 2y - \frac{49}{12} = 0$

(e) $\frac{2}{3}x - \frac{1}{6}y - 8 = 0$ (f) $3x - \frac{1}{5}y - \frac{7}{2} = 0$

7. Draw the graph of each by the *y-intercept-slope* method. (Rewrite each in the $y = mx + b$ form first.)

(a) $\{(x, y) \mid 2x + 4y + 5 = 0\}$

(b) $\{(x, y) \mid x + 4y + 8 = 0\}$

(c) $\{(x, y) \mid 4x + 8y + 15 = 0\}$

(d) $\{(x, y) \mid 3x + 7y + 12 = 0\}$

8. Draw the graph of each by the *table of values* method.

(a) $\{(x, y) \mid 2x + y + 3 = 0\}$

(b) $\{(x, y) \mid x + 3y + 6 = 0\}$

(c) $\{(x, y) \mid 3x + 2y - 6 = 0\}$

(d) $\{(x, y) \mid -4x - 2y + 7 = 0\}$

9. The relation for the speed of sound (s) in feet per second in air and the temperature (T) in Celsius degrees is given by:

$V = \{(T, s) \mid -2T + s - 1089 = 0,$
$-40 \leq T \leq 100, T \in R\}.$

(a) Rewrite the relation in the form $y = mx + b$.

(b) What is the domain of the relation?

(c) Determine the range of the relation.

(d) Is the relation a function? Explain.

(e) Graph the relation.

(f) From the graph evaluate:

(i) $V(-30)$
(ii) $V(40)$
(iii) $V(80)$.

(g) From the graph determine T:

(i) $V(T) = 1050$
(ii) $V(T) = 1100$
(iii) $V(T) = 1200$.

Is there a relation between this RANGE and the mathematical RANGE?

2·3 A New Use For Slope

The slope of a line is given by

$$m = \frac{\Delta y}{\Delta x}$$

Using the information in the graph of Figure 2-4, we find the slope of $\overleftrightarrow{AB}$.

$$m = \frac{\Delta y}{\Delta x}$$
$$= \frac{8 - 3}{6 - 2}$$
$$= \frac{5}{4}$$

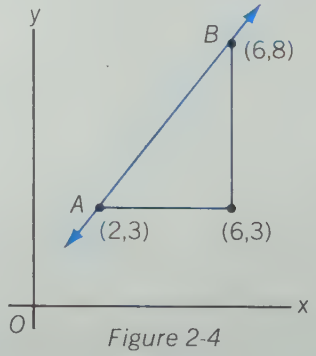


Figure 2-4

Notice that we can find the slope in this manner, too.

$$m = \frac{\Delta y}{\Delta x}$$
$$= \frac{3 - 8}{2 - 6}$$
$$= \frac{-5}{-4}$$
$$= \frac{5}{4}$$

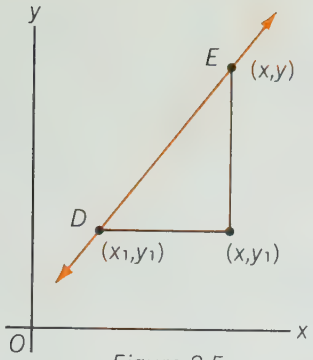


Figure 2-5

Now refer to Figure 2-5 and find the slope of the line DE in general terms.

$$\text{Slope} = m$$

$$m = \frac{y - y_1}{x - x_1} \text{ or } m = \frac{y_1 - y}{x_1 - x}$$

For convenience we will use $m = \frac{y - y_1}{x - x_1}$ in the form

$$y - y_1 = m(x - x_1).$$

This form may be used to find the equation of a line given the slope m and a point (x_1,y_1) on the line.

> $y - y_1 = m(x - x_1)$ is the point-slope form of the equation of a line l with a slope m and a point on the line $P(x_1,y_1)$.

We will now find the equation of a line when we know the slope and one point on the line.

Another use for slope.

Example 1

Determine the linear function defining the line which has a slope of $\frac{2}{5}$ and contains the point (3,2).

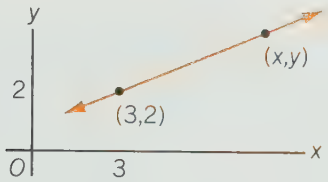


Figure 2-6

Solution

Refer to Figure 2-6. We have shown a line containing (3,2) with a slope of $\frac{2}{5}$. We have labelled a point on the line (x,y). Using the form above for slope we have:

$$m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1)$$

Method 1

$$m = \frac{y - y_1}{x - x_1}$$
$$(x_1,y_1) = (3,2)$$
$$\frac{2}{5} = \frac{y - 2}{x - 3}$$
$$(2x - 3) = 5(y - 2)$$
$$2x - 6 = 5y - 10$$
$$2x - 5y + 4 = 0$$

Method 2

$$y - y_1 = m(x - x_1)$$
$$y - 2 = \tfrac{2}{5}(x - 3)$$
$$y - 2 = \tfrac{2}{5}x - \tfrac{6}{5}$$
$$\tfrac{2}{5}x - y + \tfrac{4}{5} = 0$$
$$5(\tfrac{2}{5}x - y + \tfrac{4}{5}) = 0$$
$$2x - 5y + 4 = 0$$

At this point we have the equation in the form $Ax + By + C = 0$. Notice that slope in this case is $m = \frac{A}{B}$. Is this coincidence? Study the exercises that you do to answer the question.

The equation in this example can be rewritten in the $y = mx + b$ form.

$$2x - 5y + 4 = 0$$
$$2x - 2x - 5y + 4 - 4 = -2x - 4$$
$$-5y = -2x - 4$$
$$\frac{-5y}{-5} = \frac{-2x - 4}{-5}$$
$$y = \tfrac{2}{5}x + \tfrac{4}{5}$$

This form tells us the slope and the y-intercept.

Two equivalent forms of the function are

$$\{(x,y) | 2x - 5y + 4 = 0\} \text{ and } \{(x,y) | y = \tfrac{2}{5}x + \tfrac{4}{5}\}.$$

Two particular cases may be noted:

(i) if the line is parallel to the x-axis then

$$\Delta y = 0.$$

Therefore, $m = \frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$

and the slope is 0. (See Figure 2-7.)

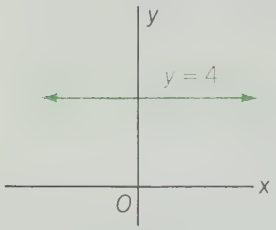


Figure 2-7

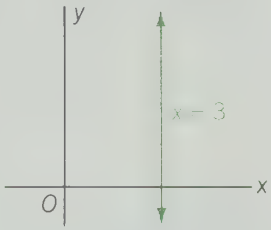


Figure 2-8

(ii) If the line is parallel to the y-axis then

$$\Delta x = 0.$$

If we make this replacement we get

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{0}.$$

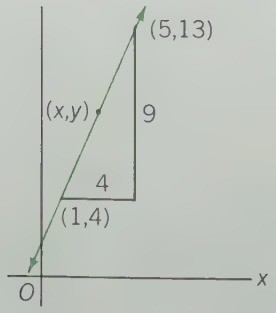
Since division by zero is undefined the symbol $\frac{\Delta y}{0}$ is undefined. Therefore we say the slope is undefined. (See Figure 2-8.)

Example 2

Determine the linear function that maps 1 into 4 and 5 into 13.

Solution

The mapping terminology is another way of saying the lines contain the points (1,4) and (5,13).



Step 1

Find the slope

$$m = \frac{13 - 4}{5 - 1}$$
$$= \frac{9}{4}$$

Step 2

Identify any point on the line (x,y).
Since slope on this line is constant we use

$$m = \frac{y - y_1}{x - x_1}$$
$$\frac{9}{4} = \frac{y - 4}{x - 1} \quad [\text{Use } (1,4) \text{ as } (x_1,y_1).]$$

Step 3

Express in appropriate form

$$9(x - 1) = 4(y - 4)$$
$$9x - 9 = 4y - 16$$

$$\boxed{9x - 4y + 7 = 0}$$

or $$\boxed{y = \tfrac{9}{4}x + \tfrac{7}{4}}$$

The linear function is

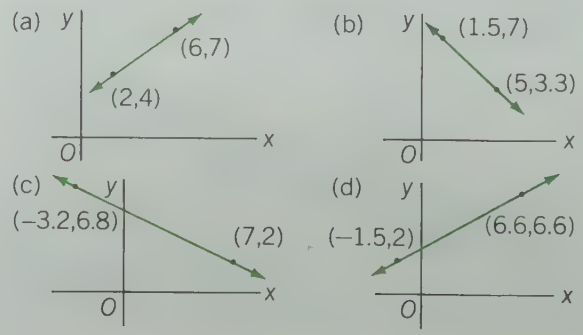
$$\{(x,y) | 9x - 4y + 7 = 0\}$$

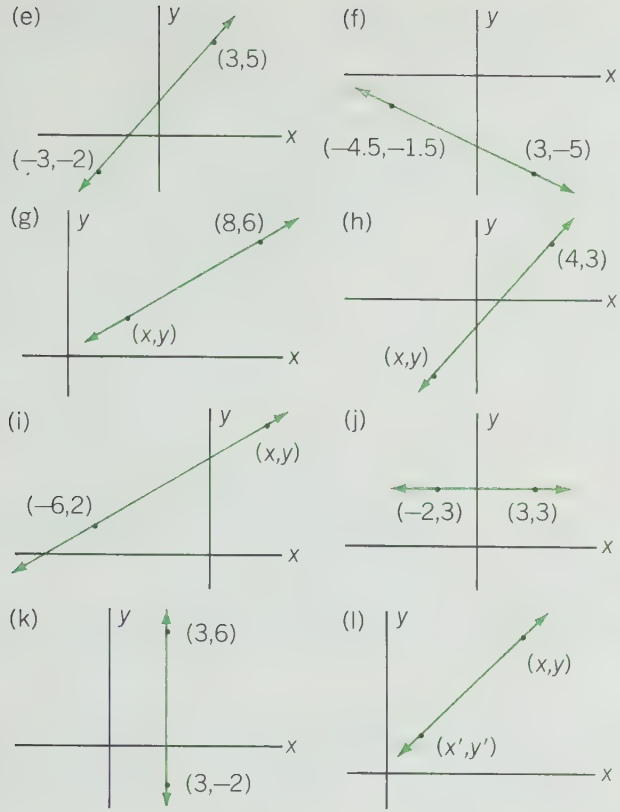
or $$\{(x,y) | y = \tfrac{9}{4}x + \tfrac{7}{4}\}.$$

The two statements are equivalent forms of the function.

EXERCISE 2·3

1. Find the slope of the line in each case:





2. Determine the linear function of the graph that contains the point (3,2) and has a slope of $\frac{5}{2}$.

3. Determine the linear function of the graph that contains the point (–5,2) and has a slope of $-\frac{4}{3}$.

4. Determine the linear function of the graph that contains the points (3,2) and (6,7).

5. Determine the linear function of the graph that contains the points (4,–6) and (–5,6).

6. Determine the linear functions of the graphs (g), (h), and (i) in question 1.

7. Refer to question 1. Determine the linear functions of the graphs in (a), (b), (c), (d), (e), and (f).

8. Refer to question 1. Determine the linear relations of the graphs in (j), (k), and (l).

2·4 Parallel and Perpendicular Lines

In Figure 2-9 four lines are drawn parallel to each other. The equation of each can be found:

(a) $y = 2x + 5$ (b) $y = 2x + 2$

(c) $y = 2x - 1$ (d) $y = 2x - 3$

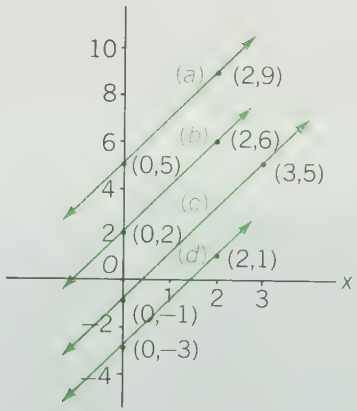


Figure 2-9

What do you notice about these lines?
What do you notice about the four equations?
Summarize what you notice about the lines and equations.

Parallel lines have the same slope.

Project

Project 1

Purpose

To find a relation between the slopes of two perpendicular lines.

Method

1. Draw the line in the Cartesian plane which contains the points $A(0,2)$ and $B(4,4)$.
2. Draw the line which contains the points $R(2,2)$ and $T(3,0)$. Notice $\overleftrightarrow{AB}$ and $\overleftrightarrow{RT}$ are perpendicular to each other.
3. Find the slope m_1 of $\overleftrightarrow{AB}$.
4. Find the slope m_2 of $\overleftrightarrow{RT}$.
5. Do you notice any relation between m_1 and m_2?
6. Repeat steps 1 through 5 for $A(1,1)$, $B(3,7)$ and $R(3,1)$, $T(6,0)$.
7. Does the relation you identified in part 5 hold for the lines in part 6? Explain.
8. In each instance find the product $m_1 \times m_2$. What conclusion do you reach?
9. Make a rule about m_1 and m_2 where m_1 and m_2 are the slopes of two lines perpendicular to each other.

Example 1

(a) Find the slope of line RV which is perpendicular to line $AB = \{(x,y) | y = \frac{2}{3}x + 5\}$.

(b) Find the equation of $\overleftrightarrow{RV}$ if it contains the point (4,6).

Solution

(a) The slope of $\overleftrightarrow{AB}$ is $\frac{2}{3}$.

Let the slope of $\overleftrightarrow{RV}$ be m_1 and the slope of $\overleftrightarrow{AB}$ be m_2. We know that since $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{RV}$, then,

$$m_1 \times m_2 = -1 \qquad \text{from Project 1}$$

hence,
$$m_1 \times \tfrac{2}{3} = -1$$
$$m_1 \times \tfrac{2}{3} \times \tfrac{3}{2} = -1 \times \tfrac{3}{2} \qquad \text{Why?}$$
$$m_1 = -\tfrac{3}{2}.$$

The slope of $\overleftrightarrow{RV}$ is $-\frac{3}{2}$.

(b) Using the slope-point form of the equation of a straight line we have

$$y - y_1 = (x - x_1)(m_1).$$

We know $m_1 = -\frac{3}{2}$ and one point is (4,6).

$$\therefore y - 6 = (x - 4)(-\tfrac{3}{2})$$
$$y - 6 = -\tfrac{3}{2}x + 6$$
$$y = -\tfrac{3}{2}x + 12$$
$$\{(x,y) | y = -\tfrac{3}{2}x + 12\}$$

EXERCISE 2·4

1. What is the slope of line AB which is parallel to line PQ whose equation is $y = 2x + 5$?

2. What is the slope of line RS which is perpendicular to line PQ in question 1?

3. Find the slope of (i) the line parallel to (ii) the line perpendicular to each of the following lines.

   (a) $\{(x,y) | y = 5x - 4,\ x,\ y \in R\}$

   (b) $\{(x,y) | y = -\frac{3}{2}x + 4,\ x, y \in R\}$

   (c) $\{(x,y) | y = -2x - 8,\ x, y \in R\}$

   (d) $\{(x,y) | y = -\frac{4}{5}x + 7,\ x, y \in R\}$

   (e) $\{(x,y) | y = \frac{6}{7}x - 3,\ x, y \in R\}$

4. Find the slope of (a) the line parallel to and (b) the line perpendicular to the line with the equation $2y + 5x + 4 = 0$.
   Hint! Transform the equation to standard form.

5. (a) Find the slope of the line BM which is perpendicular to the line containing the points $H(2, 1)$ and $K(-3, -4)$.

   (b) If $\overleftrightarrow{BM}$ contains the point (4,5) find the equation of $\overleftrightarrow{BM}$.

6. Find the slope and equation of the line AC which is perpendicular to the line containing the points $T(2,-8)$ and $S(-3,7)$ if line AC contains the point (4,4).

2·5 An Algebraic View of Direct Variation

If we wish to speak of a direct variation between two variables y and x, we may write

$$y \propto x$$

which can be read

y varies as x

y varies directly as x

y is proportional to x

y is directly proportional to x.

Translated into an algebraic equation, $y \propto x$ is $y = kx$ (k, a constant $\neq 0$). The k is called the *constant of variation* or the *variation constant* or the *proportionality constant*. This means that regardless of any changes that may take place in the two variables y and x, their ratio remains constant, i.e.,

$$\frac{y}{x} = k$$

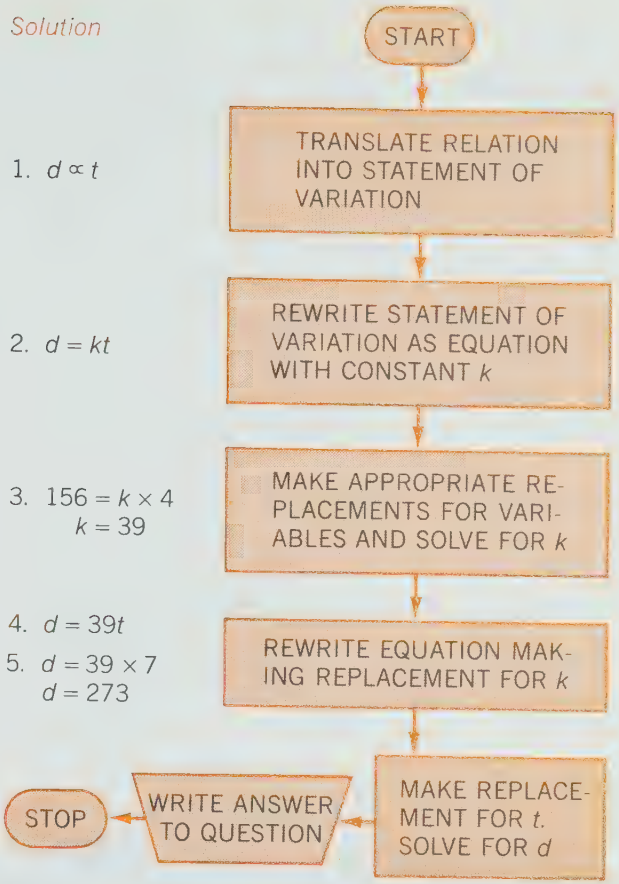
The constant of variation can be determined if we are given a value of x, $x \neq 0$ with a corresponding value of y.

Example 1

Distance travelled d varies directly as time t. Determine the constant of variation when $t = 4$ and $d = 156$.

Solution

1. $d \propto t$
2. $d = kt$
3. $156 = k \times 4$
   $k = 39$
4. $d = 39t$
5. $d = 39 \times 7$
   $d = 273$


START
TRANSLATE RELATION INTO STATEMENT OF VARIATION
REWRITE STATEMENT OF VARIATION AS EQUATION WITH CONSTANT k
MAKE APPROPRIATE REPLACEMENTS FOR VARIABLES AND SOLVE FOR k
REWRITE EQUATION MAKING REPLACEMENT FOR k
MAKE REPLACEMENT FOR t. SOLVE FOR d
WRITE ANSWER TO QUESTION
STOP

The constant of variation is 39. The function is

$$\{(t, d) \mid d = 39t\}.$$

When t is 7 the corresponding d is 273.

depth – pressure

time – distance to fall

In this section we will deal with relations. When you have completed this chapter describe each variation.

wind speed – danger

speed – time

Example 2

The distance s fallen from rest by a body in free fall is proportional to the square of the time t of the fall.

(a) If a body falls 64 feet in 2 seconds, how far will it fall in 4 seconds?

(b) Draw the graph to show the relation of s and t^2.

Solution

(a) Refer to steps in Example 1.

Step 1. $s \propto t^2$

Step 2. $s = kt^2$

Step 3. $64 = k \cdot (2 \cdot 2)$

$k = 16$

Step 4. $s = 16t^2$

Step 5. $s = 16 \times 4^2$

$s = 256$

Step 6. The body falls 256 feet in 4 seconds.

(b) We will use the *y-intercept-slope* method. Remember the two variables are s and t^2;

not t but t^2!

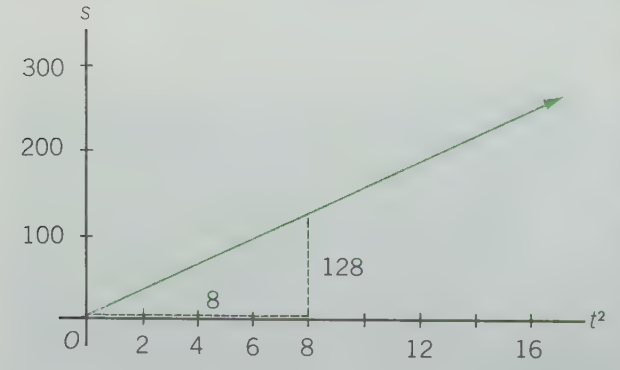
$s = 16t^2$

slope $= 16$

y-intercept $= 0$

If t^2 increases by 8, then s increases by:

$8 \times 16 = 128$.



It is important that we note the two variables in Example 2 are s and t^2, not s and t. The graph showing the relation of s and t^2 is linear whereas the graph relating s and t for $s = 16t^2$ is nonlinear.

t	0	1	2	3	4	5
s	0	16	64	144	256	400

Figure 2-10 shows the variation of s with respect to t.

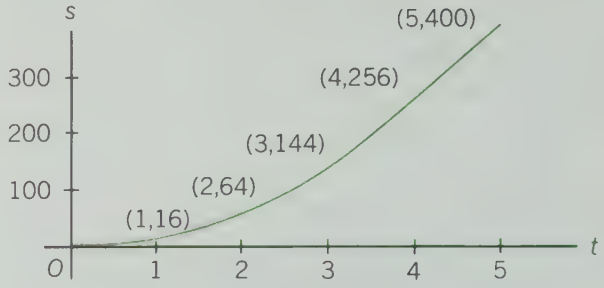
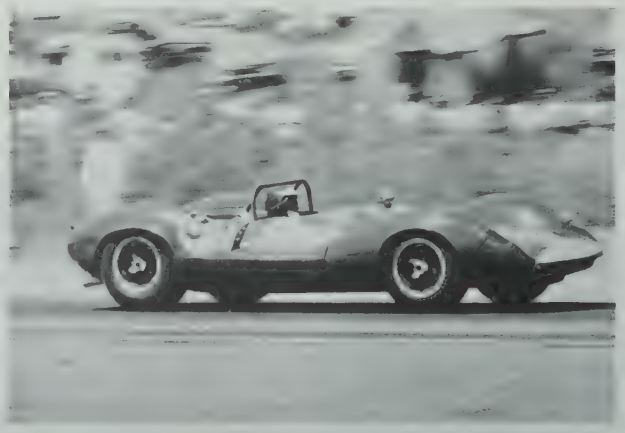


Figure 2-10

Example 3

The wind pressure P in lbs/sq. ft. on a windshield of a moving car varies as the square of the speed v in mph. If the speeds of two cars are in the ratio 2:3, compare the wind pressure on the first windshield to that of the second.



Solution 1

The student must *not* assume the speeds of the cars are 2 and 3 mph respectively. Let P_1 be the pressure in lbs/sq. ft. and $v_1 = 2m$ be the speed in miles per hour of the first car. Similarly, let P_2 be the pressure and $v_2 = 3m$ be the speed of the second car.

Step 1. Write the variation. $P \propto v^2$

Step 2. Write the equation. $P = kv^2$

Step 3. Make replacements and solve for k in Case 1. $P_1 = k(2m)^2$

$$k = \frac{P_1}{4m^2}$$

Step 4. Substitute value of k from Step 3 into the equation in Step 2.

$$P = \frac{P_1}{4m^2} v^2$$

Step 5. Make replacements for Case 2 and simplify.

$$P_2 = \frac{P_1}{4m^2} (3m)^2$$

$$= \frac{P_1}{4m^2} 9m^2$$

Step 6. Rearrange to get required comparison.

$$P_2 = \frac{9}{4} P_1$$

$$P_1 = \frac{4}{9} P_2$$

The pressure on the first is $\frac{4}{9} P_2$.

Solution 2

We will set up a proportion.

Step 1. Write the variation. $P \propto v^2$

Step 2. Write the equation. $P = kv^2$

Step 3. Make replacements for Case 1. (1) $P_1 = kv_1^2$

Step 4. Make replacements for Case 2. (2) $P_2 = kv_2^2$

Now by the property of Division we divide the members of equation (1) by the members of equation (2).

Step 5. Set up proportion.

$$\frac{P_1}{P_2} = \frac{kv_1^2}{kv_2^2}$$

Step 6. Simplify proportion.

$$\frac{P_1}{P_2} = \frac{v_1^2}{v_2^2}$$

$$= \left(\frac{v_1}{v_2}\right)^2$$

Step 7. Make replacements for the ratio of the two values of the variable and simplify.

$$\frac{P_1}{P_2} = \left(\frac{2}{3}\right)^2$$

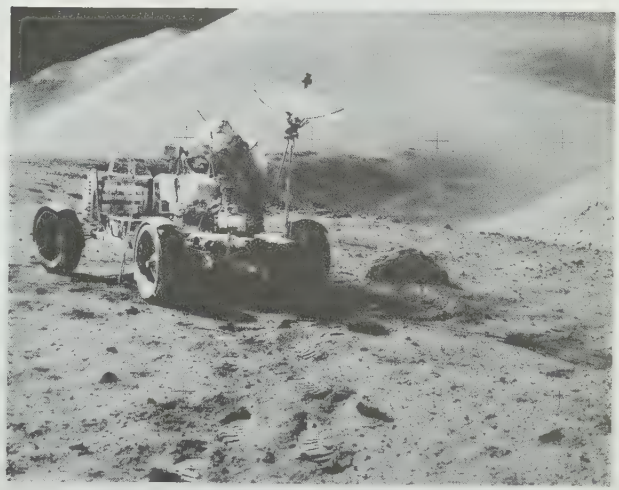
$$= \frac{4}{9}$$

The pressure on the first to that on the second is in the ratio of 4 to 9.

EXERCISE 2·5

In the problems in questions 1 to 9, translate the statement to form a suitable
(a) variation,
(b) equation.

1. The cost C of a vacation varies directly with the number of days d of vacation.
2. The stretch l of a spring is proportional to the weight w applied.
3. Surface area A of a basketball varies directly as the square of the diameter d.
4. Volume V of a sphere varies directly as the cube of the diamter d.
5. The time t required for one full swing back and forth varies as the square root of the length l of the pendulum rod.
6. The length l of a metal bar varies with the temperature t.
7. The weight W of an astronaut on another planet varies with the force of gravity g for that planet.



8. The lift L of an airplane wing varies as the square of the speed of the airplane.
9. The number of watts W of electrical power used by an appliance varies as the square of the number of amperes I used.

Study the flow chart for the type of problem illustrated in Example 1. Use this flow chart to do the next 6 questions.

10. The length of the shadow cast by a tree at noon varies as the height of the tree. A tree 22 feet high casts a shadow 12 feet long. Find the length of the shadow of a tree 54 feet high.
Hint! The length l of the shadow varies directly as the height h.

11. Refer to question 11. Find the height of the tree which casts a shadow 32 feet long.

12. The weight of a cube varies as the cube of the length of a side. The weight of a cube with a 3-centimetre side is 67.5 grams. Find the weight of a cube with 5 centimetres on each side.

13. The weight W per foot of a circular pipe varies as the square of the diameter d. If a pipe of 3-inch diameter weighs 5 pounds, find the weight of a pipe of 9-inch diameter.

14. The heat output C in calories of a circuit varies as the resistance R in ohms, all other factors remaining constant.
    (a) If 8460 calories are produced in a circuit with a 64-ohm resistance, find the heat produced by a resistance of 90 ohms.
    (b) Find the resistance required to produce 6000 calories.

15. The heat in calories C output of a circuit varies directly as the square of the current I in amperes. When the heat output is 3200 calories the current is 3.2 amperes. Find the amperage required for a heat output of 6400 calories. (Assume all other factors remain constant.)

16. Write a flow chart for the method in Example 3 which you prefer. Use the flow chart to do question 17.

17. The volume of a sphere varies directly as the cube of the radius. The radii of two spheres are in the ratio 6:11. Compare the volume of the first sphere to that of the second.

2·6 Partial Variation

The cost of renting a car is normally based on a fixed per day rate and a per mile rate. To rent a car for one day costs \$4.00 and 4 cents per mile. If a person rents a car and only drives it around the corner to his office, then leaves it there for the day (no mileage), it still costs him \$4.00. If, on the other hand, he drives the car 60 miles, he pays \$4.00 plus \$2.40 (60 × .04) or \$6.40. The total cost is based on a constant and a quantity that varies directly as the distance.

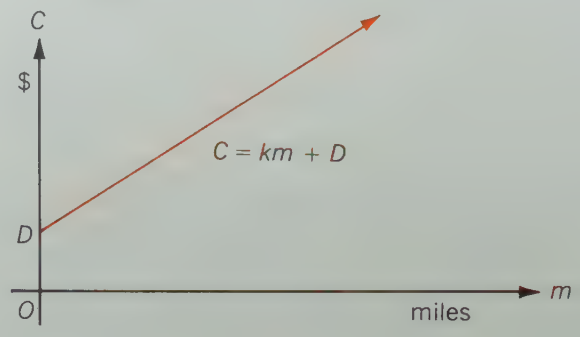
Algebraically, we indicate the cost C is made up of the flat rate D plus the mileage cost km.

$$C = D + km$$

This equation is of the form

$$y = mx + b.$$

D is the constant quantity and k is the variation constant. Graphically, D is shown as the y-intercept and k is the slope.



Example 1

The cost of renting a car consists of a fixed per day cost and a cost that varies directly as the number of miles.

(a) Write the statement of partial variation.

(b) If the cost of driving 50 miles is \$5.75, and the cost for 100 miles is \$7.50, find the per day cost and the per mile cost.

(c) Find the cost of driving 500 miles.

Solution

(1) Let the total cost be C dollars.
Let the per day cost be D dollars.
Let the number of miles driven be m miles.
Let k be the constant of variation in the part that varies.

(2) $$C = D + km$$

From the data we obtain:

(3) $$5.75 = D + 50\,k$$
(4) $$7.50 = D + 100\,k.$$

(5) (3) becomes $D = 50\,k - 5.75$.
(4) becomes $D = 100k - 7.50$.

(6) $$100k - 7.50 = 50k - 5.75$$
$$50k = 1.75$$
$$k = .035$$

(7) Substituting in (3):

$$5.75 = D + 50 \times .035$$
$$D = 4.$$

(8) Formula (2) is now:

$$C = 4 + .035m.$$

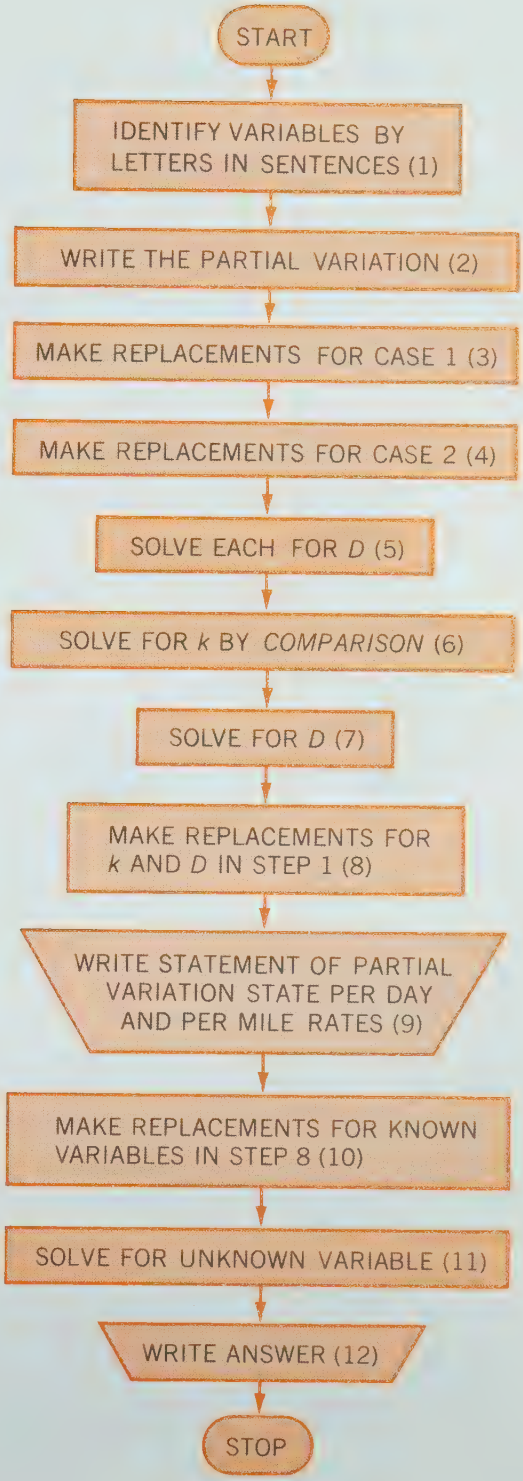
(9) (a) The statement of partial variation is:

$$C = D + km$$
$$C = 4 + .035m.$$

(b) The per day rate is \$4 and the per mile rate is 3.5¢.

(10) $$C = 4 + .035\,(500).$$
(11) $$= 4 + 17.50$$
$$= 21.50$$

(12) (c) To drive 500 miles costs \$21.50.


START
IDENTIFY VARIABLES BY LETTERS IN SENTENCES (1)
WRITE THE PARTIAL VARIATION (2)
MAKE REPLACEMENTS FOR CASE 1 (3)
MAKE REPLACEMENTS FOR CASE 2 (4)
SOLVE EACH FOR D (5)
SOLVE FOR k BY COMPARISON (6)
SOLVE FOR D (7)
MAKE REPLACEMENTS FOR k AND D IN STEP 1 (8)
WRITE STATEMENT OF PARTIAL VARIATION STATE PER DAY AND PER MILE RATES (9)
MAKE REPLACEMENTS FOR KNOWN VARIABLES IN STEP 8 (10)
SOLVE FOR UNKNOWN VARIABLE (11)
WRITE ANSWER (12)
STOP

EXERCISE 2·6

Write the statement of partial variation for each of questions 1 to 6.

1. The cost of printing a newsletter C is the sum of a fixed amount A and a quantity that varies as the number of copies P.

2. The labour cost C of repairing a TV is based on a service call charge S and partly varies as the number of hours H of labour.

3. The cost C of a banquet is based on a flat charge P and an amount that varies with the number of people attending n.

4. The annual cost of operating a car C involves a fixed cost L (licence, insurance, depreciation) and one that varies with the number of miles driven m.

5. The value of land V in a mountain resort is partly constant P and partly varies with the number of cottages at the resort n.

6. The amount John owes on his bank loan is partly constant (amount of loan) and partly varies as the time he owes the money (interest). At the end of 2 years John owes \$172.80 and at the end of 5 years he owes \$192. Assuming the interest is simple interest, find the amount of the loan and the amount he owes after 10 years.

7. Write a flow chart for the solution of the type of problem illustrated by question 6.

8. A salesman receives a fixed salary plus a commission based on sales. His salary one month was \$665 when his sales amounted to \$1300. Another month, when his sales totalled \$2600 his salary was \$730. Find his fixed salary, his commission rate and the total salary he would receive if his sales amounted to \$3450.

9. Draw the graph showing the total salary of the salesman in question 8 for sales up to \$5000.

10. The speed V of a rocket fired vertically is partly dependent on its initial velocity u and decreases directly as the square of the time t. After 2 seconds the velocity is 1232 ft/sec and after 4 seconds it is 1040 ft/sec. Find its initial speed and its speed after:

    (a) 5 seconds,

    (b) 10 seconds.

    Can you account for the *strange* answer you got to (b)?

11. Draw the graph of the relation in question 10 for $0 \leq t \leq 10$ seconds.

    (a) From the graph determine after what length of time the rocket reaches its maximum height.

    (b) Calculate after what length of time the rocket reaches its maximum height.
    Hint! What is the velocity of the rocket as it reaches its maximum height—just before it starts to descend?

How do you think the amount of pollution at a holiday resort varies with the number of permanent residents and the number of tourists?

Using the Library

1. *The Giant Golden Book of Mathematics* Read pages 67, 68. Cut out an odd-shaped piece of cardboard. Find its area by the method described on page 67.

2. *The Mathmen* Read pages 9-16. What was significant about the Rope-stretchers of ancient Egypt?

3. *Mathematics Enrichment* Ripley and Tait Copp Clark. Read pages 2, 5, 8, 11, 13, 15. Use your library to find out more about each mathematician. Which do you think contributed the most to mathematics? Why?

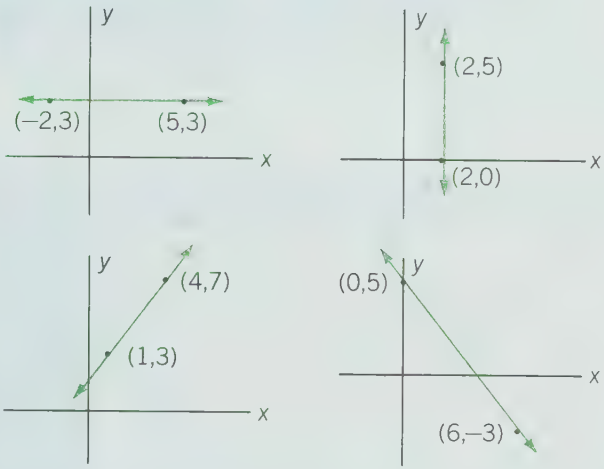
Maria Agnesi (1718-1799). Italian woman, mathematician, linguist, philosopher, and somnambulist. Another mathematician, Fermat, named a curve after her: the "witch of Agnesi". Use your library to find out more about these mathematicians.

REVIEW EXERCISE 2

1. Write in the standard $y = mx + b$ form: $3x - 5y = 4$.

2. Draw the graph of $L = \{(x,y) | 2x + 3y - 5 = 0\}$.

3. Find the slope of each:



4. Determine the linear function that contains the two points (3,1) and (5,−4).

5. Explain the relation between the slopes of:
   (a) two parallel lines
   (b) two lines perpendicular to each other.

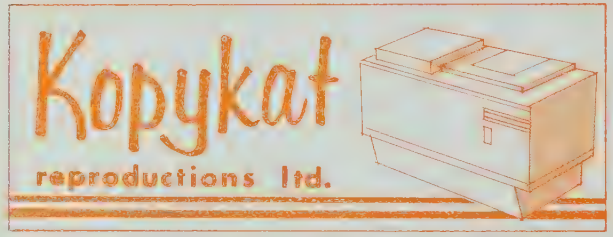
6. "The cost of cloth for a dress varies directly as the number of yards." Write this statement as
   (a) a variation (b) an equation.

7. When a body falls from rest, the number of feet it falls is proportional to the square of the time. One body falls 4 times longer than the second. Compare the distance the first body falls to that of the second.

8. The total expense e of a wedding reception involves a cost which is constant c and one which varies with the number of guests n. A reception for 25 guests costs \$175 and for 35 guests costs \$225. Find the fixed cost, cost per guest and the total expense for 50 guests.

9. The period of a pendulum t (time required for one complete swing forward and back) varies directly as the square root of its length l. The ratio of the lengths of two pendulums is 4:16. Find the ratio of the term of the first pendulum to that of the second.

10. The vibration frequency of a guitar string varies directly as the square root of the tension. The ratio of the frequency of the first string to the second string is 4:5. Compare the tensions of the two strings.



Our Company does all types of reproductions involving film, paper, cloth, and other materials. All pricing is done by the square footage of materials involved in the job. Jobs can be large or small.

We are classified as a manufacturer, so, therefore, a Manufacturers Sales Tax is applicable on all our work, unless an exemption through licence is produced.

Definitions:

Original	The material we receive from which further reproductions are made.
Square Footage	Determined by multiplying the length by the width times the number of prints required.
Reduction	Making a print smaller than the original.
Enlargement	Making a print larger than the original.
Half Size	Refers to area. To find: $L \times W =$ sq. in. $\frac{\text{Square root of square inches}}{2} =$ half size
Half Scale	Refers to linear dimension. To find: $\frac{1}{2} L$ and $\frac{1}{2} W =$ half scale

1. Customer requires 10 driprints of each of 24 original tracings, size 24″ × 36″. Price is based upon formula outlined below.

   9¢ per square foot for first 100 square feet.
   8¢ per square foot for next 400 square feet.
   7¢ per square foot for next 500 square feet.
   6¢ for all over 1000 square feet.

   A 12% Government Manufacturer's Sales Tax is applicable on the above material.

   Find total amount to be billed to customer, showing Government Sales Tax separately.

2. Customer requires 1 negative print of each of 2 drawings, size 24″ × 36″, reduced to half scale (not half size). From each of these reduced negatives, the customer requires 12 velox prints.

   Cost of negatives = $2.50 per sq. ft.
   Cost of velox prints = .95 per sq. ft.
   Sales Tax of 12% applicable.

3. Customer requires reproductions of 4 pages of Offset reproduction. Require cost for quantities of 100, 400 and 1000. Cover page is extra and requires $12.00 labour in Set-up and Art Department time. Offset plates (one required for each page) cost $4.75 each.

   100 copies – paper and printing $2.90
   400 copies – paper and printing $6.00
   1000 copies – paper and printing $11.20.

   Find cost, Sales Tax exempt.

Chapter Three
TRIGONOMETRIC FUNCTIONS

3·1 The Trigonometric Angle

The teacher and student are referred to Book 2 for a number of projects involving practical field work. If these have not been done previously Book 2 should be obtained and a selection made from the projects in Chapter 10.

Section 3·4 relating to the trigonometric functions of θ, $90° < \theta < 360°$ is essential for later work in Chapter 15: *Solution of Oblique Triangles*.

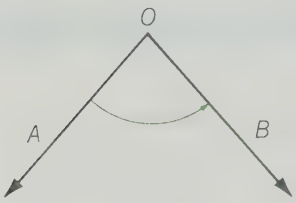


Figure 3-1

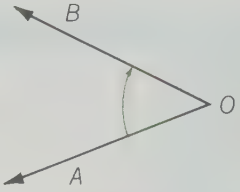


Figure 3-2

In this section, let us consider the following trigonometric representation of an angle. We will use two rays, $\overrightarrow{OA}$ and $\overrightarrow{OB}$, with the common endpoint O such that one ray is fixed and the other ray is free to rotate about the point O. The fixed ray is the *initial ray* and the movable ray is the *terminal ray*. The direction of the rotation is indicated by an arrow affixed to an arc. In Figure 3-1 the terminal ray OB has been rotated counterclockwise from the initial ray OA. In Figure 3-2 the terminal ray OB has been rotated clockwise from the initial ray OA.

An angle is determined if and only if the following three items of information, or their equivalent, are given:

1. the position of the initial ray,
2. the position of the terminal ray,
3. the direction of the rotation.

An angle is in *standard position* in the coordinate plane when:

1. the point of rotation is the origin,
2. the initial ray is the positive portion of the x-axis,
3. the rotation of the terminal ray is in a counterclockwise direction.

In Figure 3-3 $\angle AOB$ is in standard position.

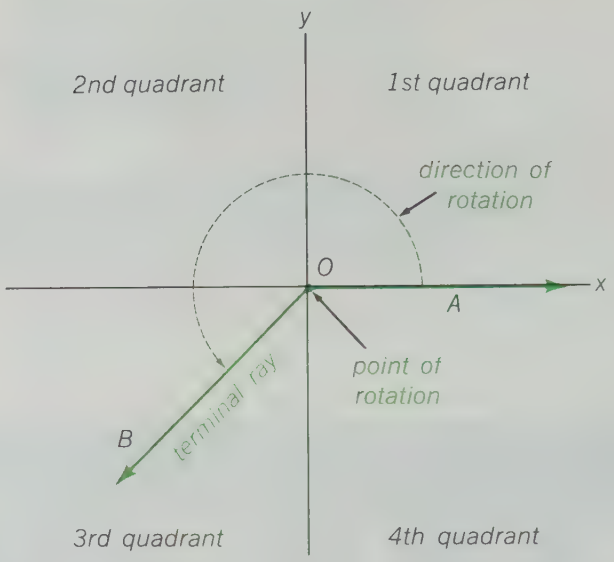


Figure 3-3

A figure skater rotates through many revolutions. What is a pirouette?

The *measure* of an angle is determined by:

1. the direction of rotation,
2. the amount of rotation.

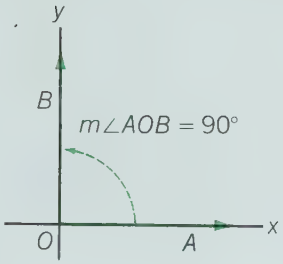


Figure 3-4

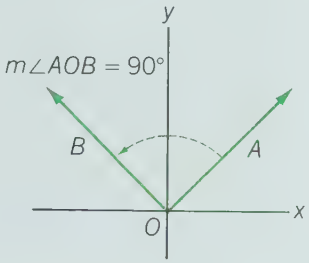
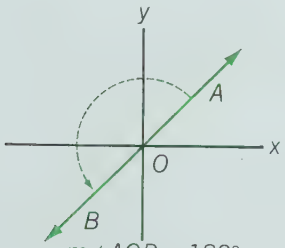
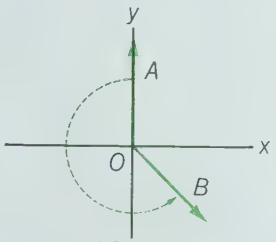


Figure 3-5



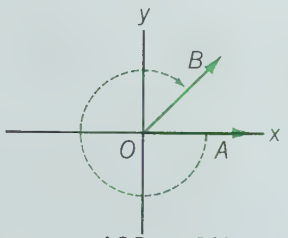
$m\angle AOB = 180°$

Figure 3-6



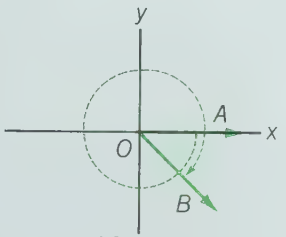
$m\angle AOB = 225°$

Figure 3-7



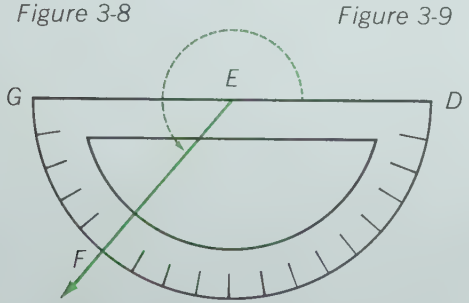
$m\angle AOB = -315°$

Figure 3-8



$m\angle AOB = -405°$

Figure 3-9



$m\angle DEF = m\angle DEG + m\angle GEF$

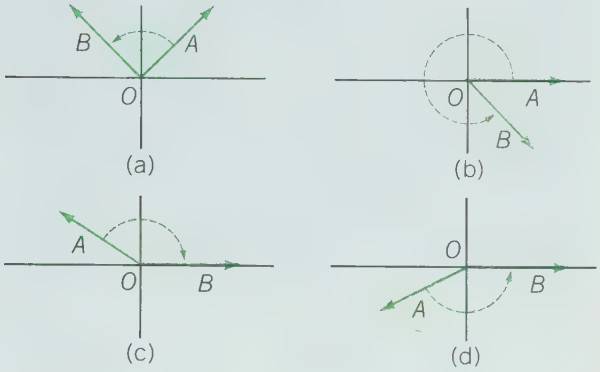
$= 180° + 50°$

$= 230°$

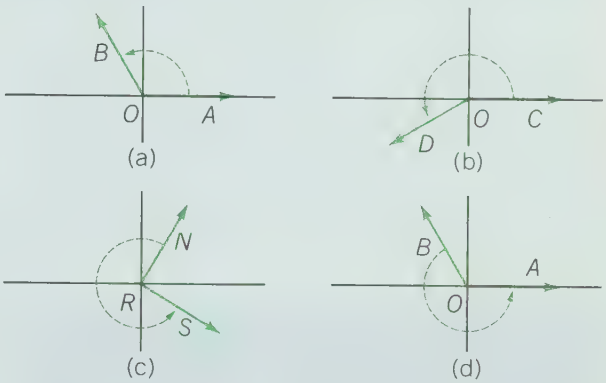
Figure 3-10

EXERCISE 3·1

1. Which of the angles are in standard position? For those that are not, tell why they are not.



2. Use a protractor to find the degree measure of the rotations in each of the trigonometric angles.



3. Draw an angle in standard position with a measure in degrees of:
   (a) 140 (b) 270 (c) 310 (d) 335 (e) 360.

4. Write in words:
   (a) $m°\angle XOY = 227$ (b) $m°\angle MON = 346$.

5. Write in symbols:
   (a) the measure in degrees of angle *AOF* is 250,
   (b) the measure in degrees of angle *ANT* is 310.

6. Look up in a dictionary the word *quadrant*. What is the root of the word? Relate the meaning given in the dictionary to this work.

3·2 Forward and Lateral Displacements

Consider a football player running down the field with the ball on the course shown in Figure 3-11. Notice that the player carried the ball a lot farther in going from *A* to *B* than he did in going from *B* to *C* yet the gain in yardage in going from *A* to *B* was less than in going from *B* to *C*.

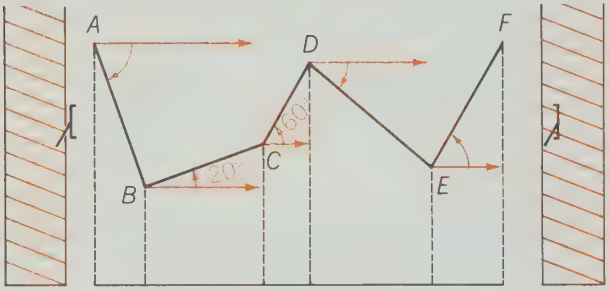


Figure 3-11

We will call the yards gained the forward displacement for that carry. Let us look at the carry from *B* to *C* more closely. This carry has been drawn on a larger scale in Figure 3-12. Using a ruler and the scale given we find the carry was 30 yards and the forward displacement was 28 yards.

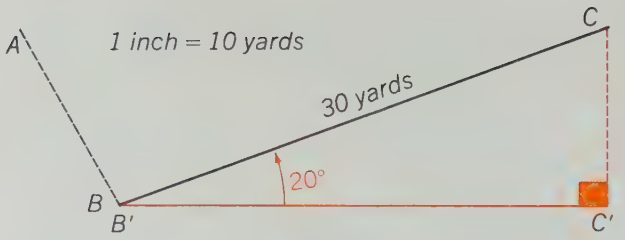


Figure 3-12

Compare the carry from *B* to *C* and its resulting yards gained with the carry from *C* to *D* and its resulting yards gained (Figure 3-13). In the latter case the carry was 30 yards also but the gain was only 15 yards.

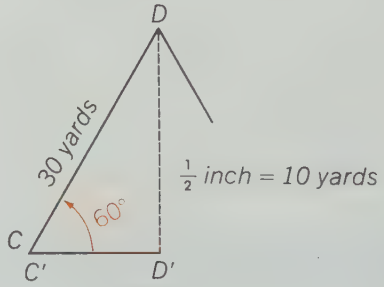


Figure 3-13

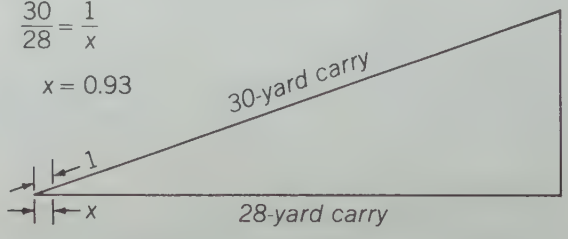
Notice that the measure of the angle in the second carry is greater than that in the first carry. It appears that the size of the angle affects the yards gained when the length of the carry is constant.

Consider the first carry again. Find the yards gained for every yard the player ran. To do this you should use a proportion obtained from similar triangles.

Let *x* represent the yards gained.

$$\frac{30}{28} = \frac{1}{x}$$

$$x = 0.93$$



Hence the player gains 0.93 yards per yard run. Now, if we know the distance the player ran we can find the yards gained, or if we know the yards gained we can find the distance he ran.

Example 1

(a) If the player ran at an angle of 20° for 42 yards how many yards did he gain?

(b) If the player gained 60 yards how far did he carry?

Solution

(a) 1 yard of carry results in 0.93 yards gain. 42 yards of carry results in $42 \times 0.93 = 39.5$ yards gain.

Alternate Method:

Let x represent the yards gained.

$$\frac{1}{42} = \frac{0.93}{x}$$

$$x = 39.5$$

(b) 1 yard of carry results in 0.93 yards gain. x yards of carry results in 60 yards gain.

$$\frac{1}{x} = \frac{0.93}{60}$$

$$x = 64.5 \text{ yards.}$$

Now refer to Figure 3-13. Calculate the yards gained per yard of carry. Did you get 0.5 yards per yard of carry?

Consider now the sideward displacement. During the carry from B to C the sideward (called lateral hereafter) displacement was 10 feet (from the scale drawing) but during the carry from C to D the lateral displacement was 26 yards (from scale drawing). It appears the measure of the angle has an effect on the lateral displacement also when the distance carried is constant. Hence, the player moves laterally 10 yards for a carry of 30 yards or 0.3 yards for 1 yard carry.

Find the lateral displacement for carry from C to D. What was the lateral displacement for each yard of carry?

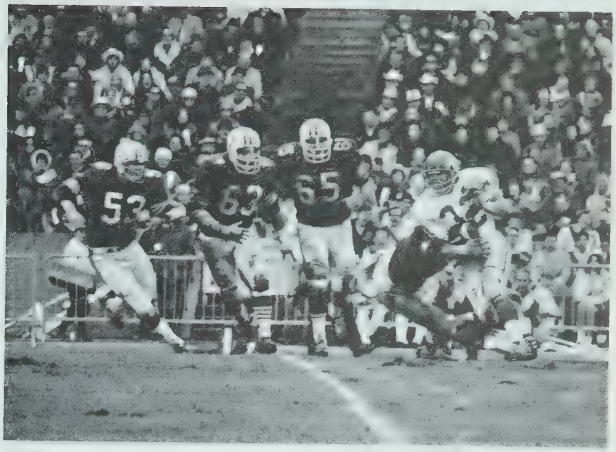
Example 2

(a) Draw a forward displacement line and a carry line at an angle of 40° to this forward displacement line.

(b) If the player carried the ball 20 yards on the *run* find the yards gained.

(c) Find the yards gained per yard of carry.

(d) Use the answer to part (c) to find the yards gained if the carry is 50 yards.



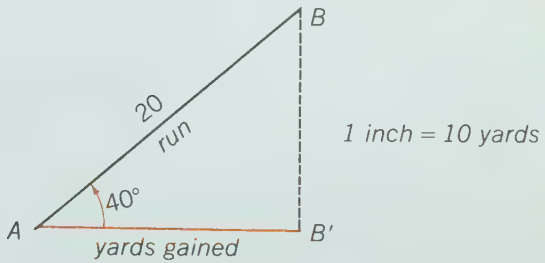
(e) Find the lateral displacement in yards per yard carried.

Solution

(a) The diagram shows the carry and the yards gained.

(b) Find the measure of $\overline{AB'}$ by using the scale drawing.

Yards gained is 15.



(c)
$$\frac{20}{15} = \frac{1}{x}$$

$$x = 0.75$$

0.75 yards gained per yard carried.

(d)
$$50 \times 0.75 = 37.5$$

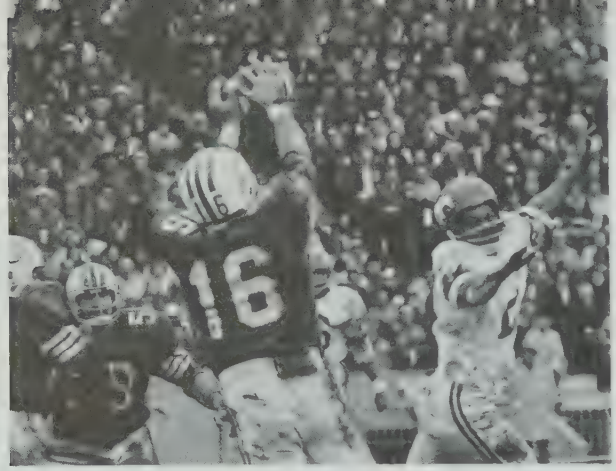
Yards gained is 37.5 yards.

(e) Using the scale diagram we find the lateral displacement is 13 yards.

$$\frac{20}{13} = \frac{1}{x}$$

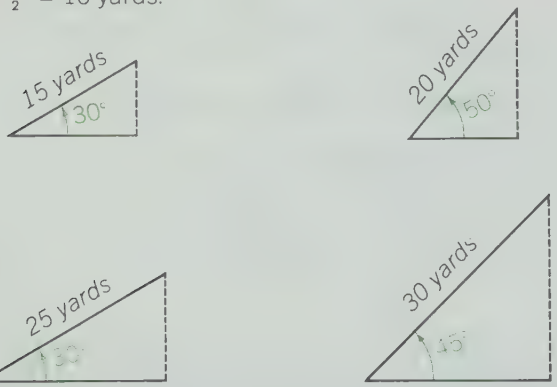
$$x = 0.65$$

The lateral displacement per yard carried is 0.65.



EXERCISE 3·2

1. The following diagrams are drawn to the scale $\frac{1}{2}'' = 10$ yards.



A football player made completed football passes in which the ball travelled the distances shown.

Find for each:
(a) the yards gained,
(b) the yards gained per yard carried,
(c) the lateral displacement,
(d) the lateral displacement per yard carried.

2. Use the information gained in question 1 to find (i) yards gained (ii) lateral displacement, in each of the following.
   (a) A player ran at an angle of 30° for 38 yards.
   (b) A player ran at an angle of 50° for 62 yards.
   (c) A player ran at an angle of 30° for 108 yards.
   (d) A player ran at an angle of 45° for 74 yards.

3. A player ran at an angle of 25° for 50 yards.
   Find
   (a) yards gained,
   (b) yards gained per yard carried,
   (c) the lateral displacement,
   (d) the lateral displacement per yard carried,
   (e) yards gained had he carried 68 yards,
   (f) yards carried had he gained 52 yards,
   (g) yards of lateral displacement had he carried 22 yards,
   (h) yards carried had there been 38 yards lateral displacement.

3·3 Using Displacement Ratios

Consider forward and lateral displacements of the directed segment *OB* in Figure 3-14. Movements from the *y*-axis to the *right* are *positive* forward displacements (positive numbers on the *x*-axis). Movements to the *left* of the *y*-axis are *negative* forward displacements. Why? Movements *upwards* from the *x*-axis are *positive* lateral displacements. Movements *downward* from the *x*-axis are *negative* lateral displacements.

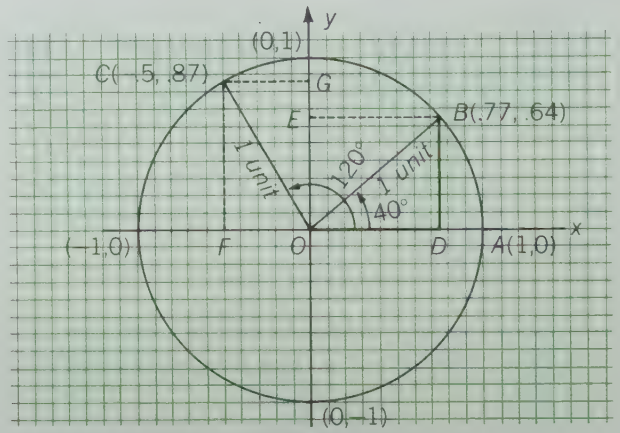


Figure 3-14

Associated with the directed segment *OB* is the forward displacement $\overline{OD}$. From Figure 3-14 we find the measure of $\overline{OD}$ is 0.77.

Similarly, the displacements associated with the directed segment OC are:

angle measure	forward	lateral	$\frac{\text{lateral displacement}}{\text{forward displacement}}$
130°	−.64	.77	$\frac{.77}{-.64} = -1.20$

Notice the forward displacement for 130° is negative. Can you explain why? Notice also in the third column the ratio of the lateral displacement to the forward displacement is given.

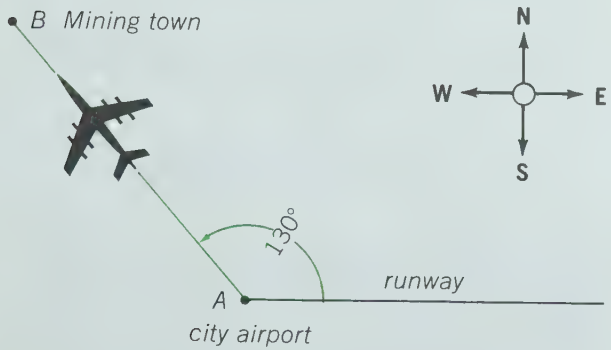
The forward displacement is given by the abscissa (x-value) on the unit circle in Figure 3-14. The lateral displacement is given by the ordinate (y-value).

Example 1

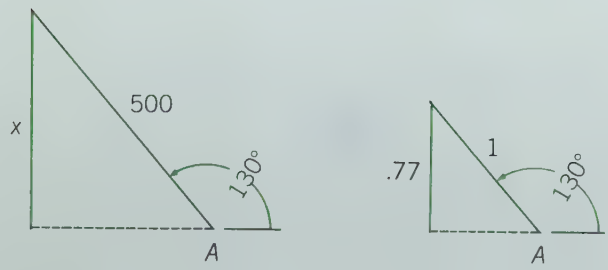
An airplane flies on the path shown. The flying distance between the airport A and the mining town B is 500 miles.

(a) How far north of the airport is the mining town?

(b) How far west of the airport is the mining town?



Solution



The lateral displacement (northward movement) associated with a segment which forms an angle of 130° is 0.77 for each unit of the segment. Therefore, by similar triangles, we have for the lateral displacement

$$\frac{500}{x} = \frac{1}{.77}$$
$$x = 500 \times .77$$
$$= 385$$

(a) Northward movement is 385 miles.

(b) Using similar triangles again, we can find the forward displacement: (Positive forward is eastward.)

$$\frac{500}{x} = \frac{1}{-.64}$$
$$x = -320.$$

(From chart above we know forward displacement for 130° is −.64.)

This means there was a *negative* 320 miles *forward* displacement or 250 miles movement westward.

Thus the movement westward was 320 miles. Therefore, the mining town is 385 miles north and 320 miles west of airport A.

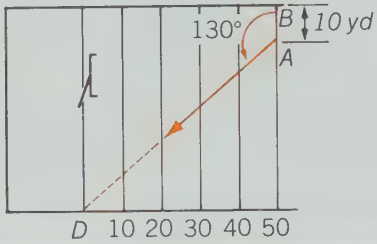
EXERCISE 3·3

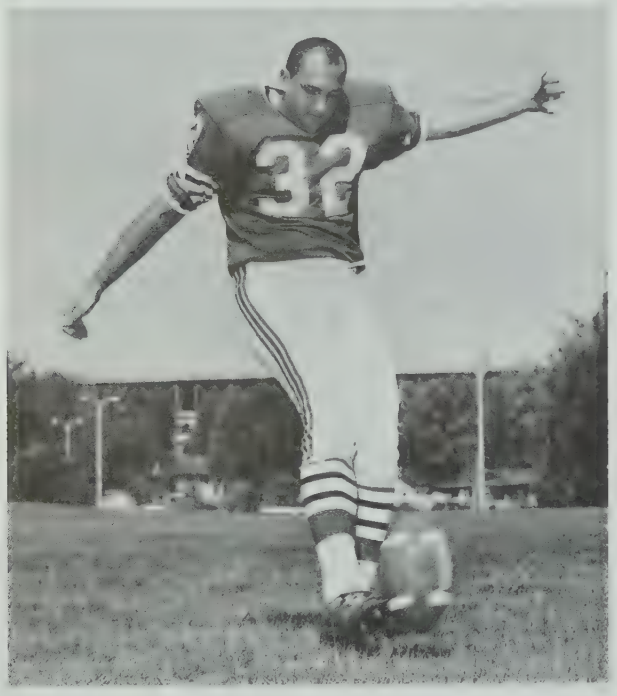
1. If 40 miles east of Town X is a positive forward displacement, what is meant by a forward displacement of −45 miles east?

2. If the lateral displacement in a physical model is interpreted as 400 feet north, how do you interpret a lateral displacement of −500 feet?

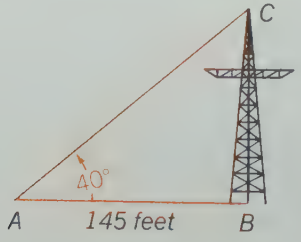
3. A schooner travels on the route shown. What is its northward displacement? its eastward displacement?

4. Football player A stands on the opposing team's 50 yard line and 10 yards in from the edge of the field. He kicks a ball at an angle as shown ($m\angle BAD = 130°$). The ball travels 60 yards. What is the *forward* displacement, or yards kicked? Does the ball cross the goal line?

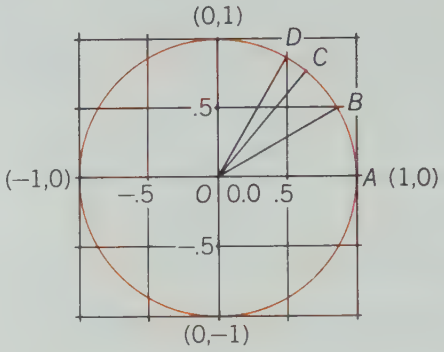




5. From the diagram shown find the length of the guy wire $\overline{AC}$ when $m(\overline{AB})$ is 145 feet.



6. In the centre of a sheet of graph paper draw a set of axes. Use the origin as centre to draw a circle with radius 2 inches. Use the scale 2 inches = 1 unit and label the diagram as shown. Use a protractor to draw in standard position $\angle AOB$, $\angle AOC$, $\angle AOD$ respectively, with measures 30°, 50° and 60°. Label the coordinates B, C, D.



7. Use the information from question 6 to do the following:

   (a) What are the forward and lateral displacements of the directed segment OB if $m(\overline{OB}) = 44$?

   (b) What are the forward and lateral displacements of the directed segment OC if $m(\overline{OC}) = 37$?

   (c) What are the forward and lateral displacements of the directed segment OD if $m(\overline{OD}) = 86$?

8. How far north and how far east does an airplane move if it flies 150 miles on a path

   (a) 30° north of east

   (b) 50° north of east

   (c) 60° north of east?

3·4 Trigonometric Functions

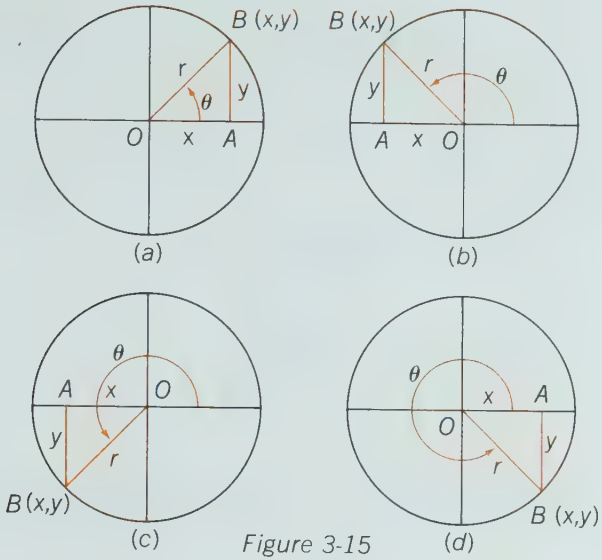


Figure 3-15

Consider Figure 3-15(a). The radius of the circle is r units. $\angle AOB$ is in standard position.

$$m(\overline{OA}) = \text{abscissa of } B = x$$
$$m(\overline{AB}) = \text{ordinate of } B = y$$
$$m(\overline{OB}) = \text{radius of circle} = r$$

Since the measure of $\angle AOB$ is not specified, the terminal ray could be in any quadrant. Diagrams (b), (c), and (d) show the terminal arm in each of the other 3 quadrants. Studying the figures we see that $\overline{OA}$ and $\overline{AB}$ are directed segments which may be positive or negative in value. $\overline{OB}$ is always understood to be positive.

We are interested in three ratios involving x, y, and r.

Definitions

$$\text{sine } \theta = \sin\theta = \frac{\text{measure of } \overline{AB}}{\text{measure of radius } \overline{OB}} = \frac{y}{r}$$

$$\text{cosine } \theta = \cos\theta = \frac{\text{measure of } \overline{OA}}{\text{measure of radius } \overline{OB}} = \frac{x}{r}$$

$$\text{tangent } \theta = \tan\theta = \frac{\text{measure of } \overline{AB}}{\text{measure of } \overline{OA}} = \frac{y}{x}$$

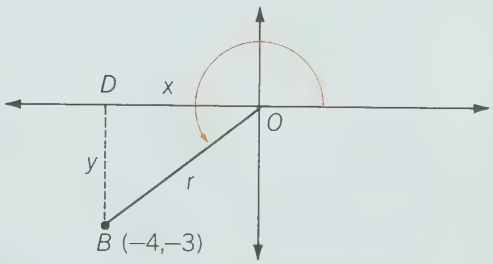
The triangle AOB in each of the four figures is called the reference triangle or triangle of reference. Study the table below for the sign of the functions for different positions of the reference triangle.

Table 1

Diagram	x	y	r	Sign of Value of $\frac{x}{r}(\cos\theta)$	Sign of Value of $\frac{y}{r}(\sin\theta)$	Sign of Value of $\frac{y}{x}(\tan\theta)$
(a)	+	+	+	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{+}{+} = +$
(b)	−	+	+	$\frac{-}{+} = -$	$\frac{+}{+} = +$	$\frac{+}{-} = -$
(c)	−	−	+	$\frac{-}{+} = -$	$\frac{-}{+} = -$	$\frac{-}{-} = +$
(d)	+	−	+	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$

Example 1

Find $\sin\theta$, $\cos\theta$, $\tan\theta$ from the diagram.



Solution

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5} \qquad \cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5} \qquad \tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$$

Note! To find r we must use the Pythagorean property:

$$a^2 + b^2 = c^2$$
$$m(\overline{OD}) = a,\ a = 3$$
$$m(\overline{DB}) = b,\ b = 4$$
$$m(\overline{OB}) = c,\ c = \ ?$$
$$3^2 + 4^2 = c^2$$
$$c^2 = 25$$
$$c = 5$$
$$c = r = 5 \text{ } (r \text{ is always positive.})$$

Hence, $r = \sqrt{a^2 + b^2}$

In labelling the sides of a right triangle it is customary to use x, y, and r when in the coordinate plane and r is always the radius or hypotenuse of the right triangle. However, when we are about to use the Pythagorean relation in a triangle we often use the letters a, b and c for the measures of the sides. In this case, c is always the measure of the hypotenuse, and a and b the measures of the other two sides.

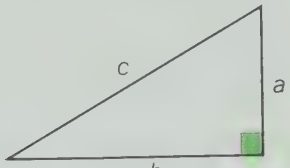


Figure 3-16

Another scheme sometimes used is illustrated in Figure 3-17. Note the angles are labelled M, N, and T. The side opposite an angle is labelled with the small case letter corresponding to the capital case letter of the angle.

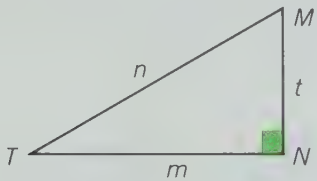
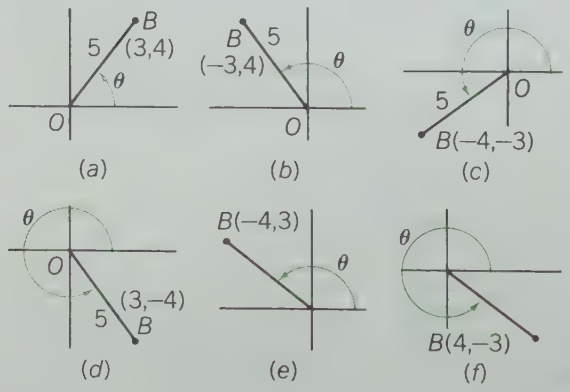


Figure 3-17

EXERCISE 3·4

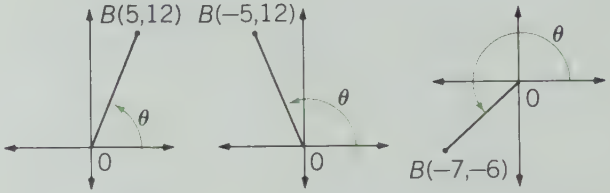
1. In each of the following find sin θ, cos θ, tan θ.



2. The terminal ray OB of an angle in standard position in a Cartesian coordinate system contains the point (4,–7). Find the value of sin θ, cos θ, tan θ.
Hint! Use $r = \sqrt{x^2 + y^2}$.

3. Given sin $\theta = \frac{2}{5}$, find cos θ and tan θ if the terminal ray is in the second quadrant.

4. Find r in each case. Then find sin θ, cos θ, tan θ for each.



3·5 The Right Triangle

In Section 3·4 we defined the three trigonometric functions in terms of the abscissa, ordinate and radius. We shall be more restrictive in this section and define the three functions in terms of a right triangle without reference to the coordinate system. The *right triangle* is a triangle with one right angle – sometimes called a *right-angled* triangle.

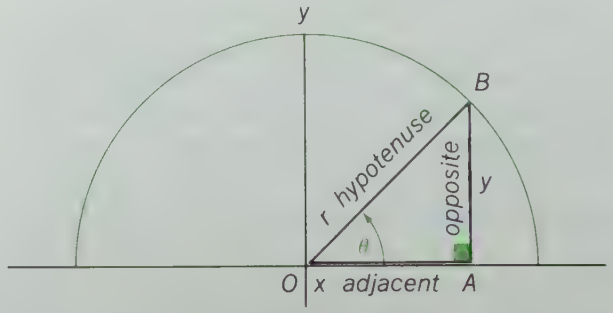


Figure 3-18

Notice in Figure 3-18:

y is the measure of $\overline{AB}$ – the side opposite angle θ
x is the measure of $\overline{OA}$ – the side adjacent to angle θ
r is the measure of $\overline{OB}$ – the hypotenuse.

Definitions

$$\sin\theta = \frac{y}{r} = \frac{\text{measure of side opposite angle } \theta}{\text{measure of hypotenuse}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{measure of side adjacent to angle } \theta}{\text{measure of hypotenuse}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{measure of side opposite angle } \theta}{\text{measure of side adjacent to angle } \theta}$$

These relations can be used with reference to any right triangle. For brevity we write the three relations

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\text{opp}}{\text{hyp}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\text{adj}}{\text{hyp}}$$

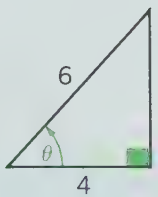
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\text{opp}}{\text{adj}}$$

When these three functions are used in this manner the measure of the angle θ is restricted.

$$0° < \theta < 90°$$

The orientation of the triangle is immaterial.

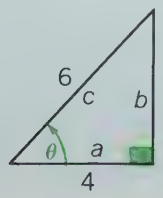
Example 1



(a) Find the third side of the triangle.
(b) Find the value of sin θ, cos θ, tan θ.

Solution

(a) *Note!* measure of adjacent side = 4
measure of hypotenuse = 6
measure of opposite side = ?



Using Pythagorean Property

$$a^2 + b^2 = c^2$$
$$16 + b^2 = 36$$
$$b^2 = 36 - 16$$
$$b^2 = 20$$
$$b = \sqrt{20}$$

Measure of opposite side = $\sqrt{20}$.

Do you remember how to simplify this? If not, we shall review how in Chapter 11. For the present, use the square root tables.

(b) $$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{20}}{6}$$

(c) $$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{6} = \frac{2}{3}$$

(d) $$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{20}}{4}$$

How could a carpenter use "trig"?

Example 2

Given $\sin\theta = \frac{3}{4}$, find $\cos\theta$ and $\tan\theta$.

Solution

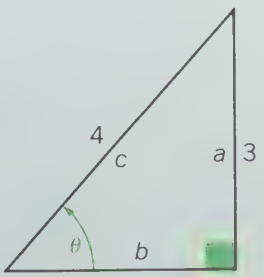
Draw a triangle, and label one angle θ, the opposite side 3 and the hypotenuse 4.

Now find the third side:

$$c^2 = a^2 + b^2$$
$$16 = 9 + b^2$$
$$b^2 = 7$$
$$b = \sqrt{7}$$

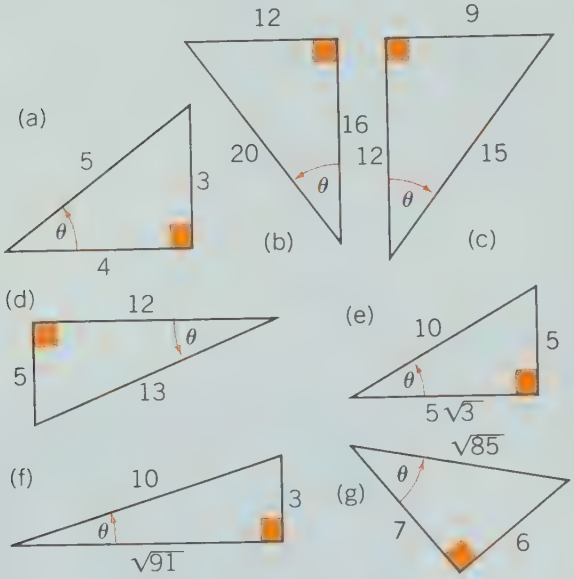
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{7}}{4}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{\sqrt{7}}$$

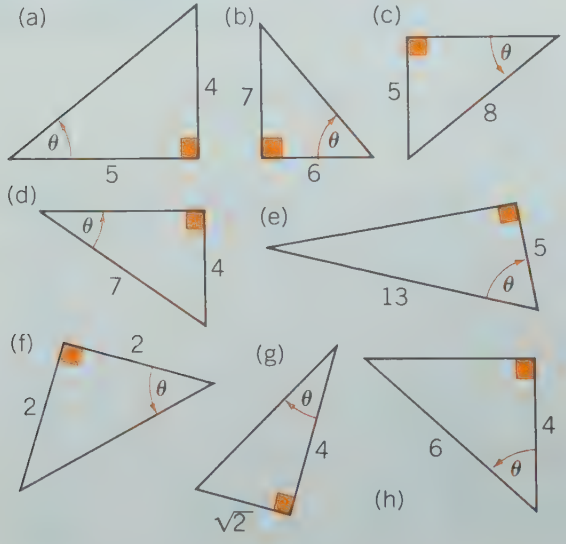


EXERCISE 3·5

1. State the values of $\sin\theta$, $\cos\theta$, $\tan\theta$ in terms of the measure of the sides of each triangle. (Express each as a fraction.)



2. Find the measure of the unknown side in each triangle, then give the value of
(i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$
for each (as a fraction).



3. Given $\sin\theta = \frac{4}{7}$, find $\cos\theta$ and $\tan\theta$.

4. Given $\tan\theta = \frac{10}{7}$, find $\sin\theta$ and $\cos\theta$.

5. Given $\cos\theta = \frac{6}{9}$, find $\sin\theta$ and $\tan\theta$.

6. Given $\sin\theta = \frac{10}{11}$, find $\cos\theta$ and $\tan\theta$.

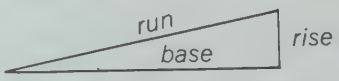
7. Given $\cos\theta = \frac{7}{12}$, find $\sin\theta$ and $\tan\theta$.

8. Given $\tan\theta = 1$, find $\sin\theta$ and $\cos\theta$.

9. Use ruler and compasses to construct a right triangle with $\sin\theta = \frac{3}{5}$. How many different right triangles can you construct to satisfy this condition? Explain.

10. A carpenter gives the pitch of a roof as $\frac{1}{2}$ foot rise per foot of run. What trigonometric relation is he using?

11. Gradient of a roadbed is given as the ratio rise to the run of the roadbed. What trigonometric ratio is the engineer using?



3·6 Sin θ, Cos θ, Tan θ, $0° \leq \theta \leq 90°$

Projects

Project 1

Purpose

(a) To study the value of sin θ as θ increases in measure from 0° to 90°.

(b) To study the value of cos θ as θ increases in measure from 0° to 90°.

(c) To study the value of tan θ as θ increases in measure from 0° to 90°.

Method

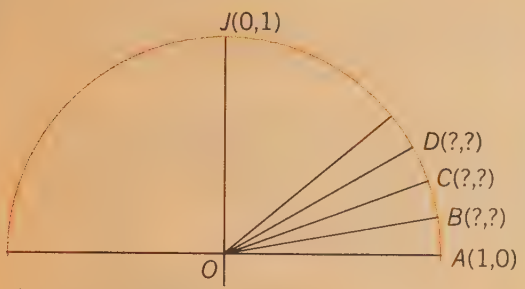


Figure 3-19

Refer to Figure 3-19.

1. On a sheet of graph paper (10 lines to the inch is the best) draw the set of coordinate axes with origin *O*.
2. With centre *O* draw a circle with radius 2 inches, (let scale be 1 unit = 2 inches) or other radius to suit graph paper.
3. Draw ∠AOB in standard position so that $m\angle AOB = 10°$.
4. Continue to draw angles *AOC*, *AOD*, etc. with measures 20°, 30°, etc. to include $\angle AOJ = 90°$.
5. As accurately as possible, find the abscissa and ordinate of each point *A*, *B*, *C*, *D*, etc. from the graph paper.
6. Record the data obtained and complete Table 1.

Table 1

angle measure	abscissa x	ordinate y	radius r	sin θ $\frac{y}{r}$	cos θ $\frac{x}{r}$	tan θ $\frac{y}{x}$
0°	1	0	1	$\frac{0}{1} = 0$	$\frac{1}{1} = 1$	$\frac{0}{1} = 0$
10°						
20°						
30°						
⋮						
80°						
90°	0	1	1	$\frac{1}{1} = 1$	$\frac{0}{1} = 0$	undefined

7. On the lower portion of the sheet of graph paper, graph the three trigonometric functions from Table 1. Use a large scale for each function except tan θ. Join the points on each graph with a smooth curve.

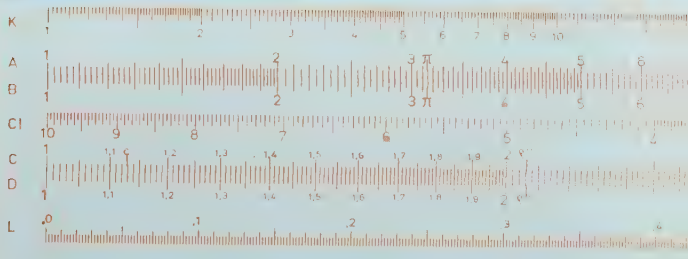
Questions

1. As θ increases in measure from 0° to 90°, how do the values of (a) sin θ, (b) cos θ, (c) tan θ change?
2. In Table 1, tan 90° is undefined. Explain.
3. Complete this table:

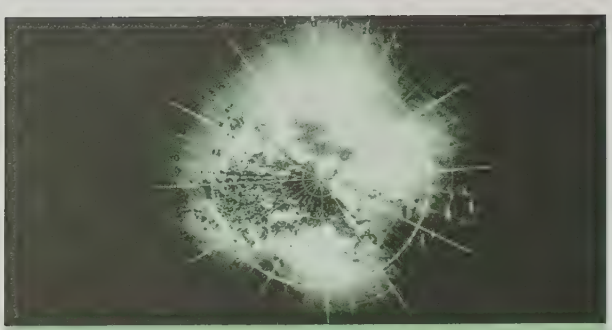
Table 2

θ	0°	10°	20°	30°	40°	50°	60°	60°	80°	90°
sin θ										
cos θ										
tan θ										

4. From Table 2 calculate an approximation for sin 75°. Now find from the graph drawn in Step 7 the value of sin 75°. Is there a difference in the two values? If there is a difference can you account for the difference? You might want to try another: tan 25°.



Your slide rule helps you to make calculations quickly and easily.



How do you think trigonometry is used in radar?

Project 2

Purpose

To determine how the value of r affects the values of the function found in Project 1.

Method

Left to the student.
Hint! Repeat parts of Project 1 using $r = 2, 3, 4$.

Questions

1. When the value of the radius increases, how does the value of $\cos\theta$ change? of $\sin\theta$? of $\tan\theta$?
2. Write a general statement answering the purpose of this Project.

3·7 Using the Tables

Now that we have a partial table of values for the three trigonometric functions, we shall use it to solve some problems. There are two types of problems we shall solve using trigonometric functions.

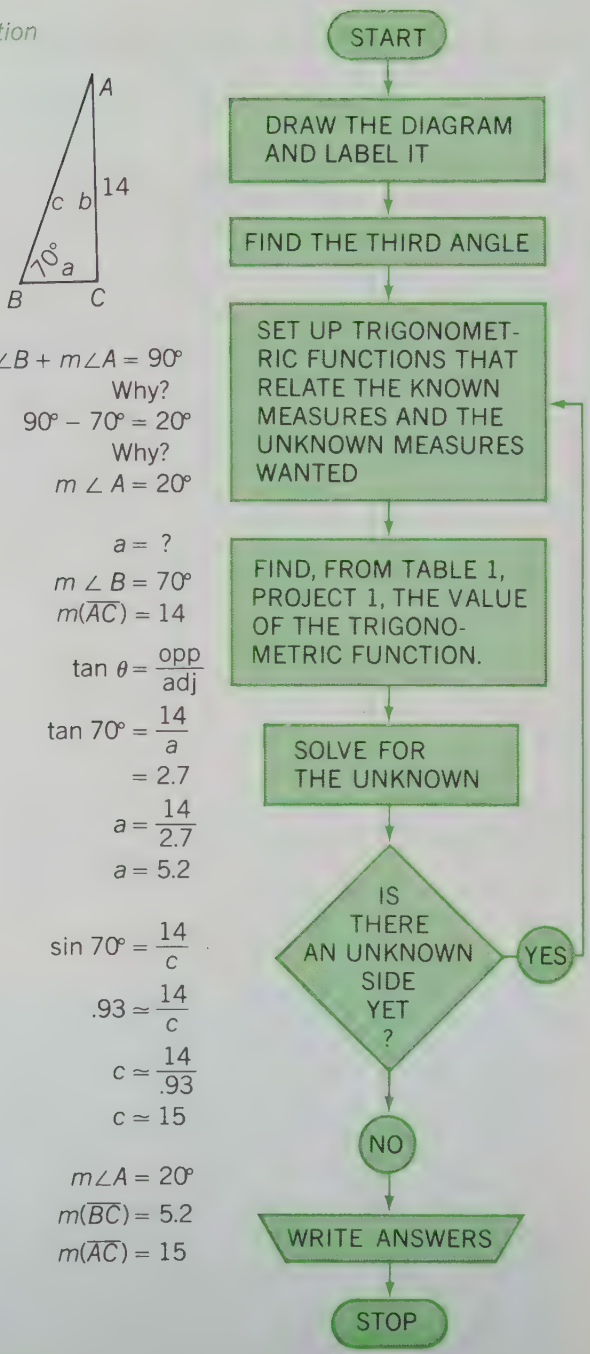
Problem 1

Given the measures of an angle (other than the right angle) and a side of a right triangle, find the measures of the other sides, and the angle. (In short, *solve* the triangle.)

Example 1

Given $\triangle ABC$ with $m\angle C = 90°$, $m(\overline{AC}) = 14$, and $m\angle ABC = 70°$, find $m\angle A$, $m(\overline{BC})$, and $m(\overline{AB})$.

Solution


START
DRAW THE DIAGRAM AND LABEL IT
FIND THE THIRD ANGLE
SET UP TRIGONOMETRIC FUNCTIONS THAT RELATE THE KNOWN MEASURES AND THE UNKNOWN MEASURES WANTED
FIND, FROM TABLE 1, PROJECT 1, THE VALUE OF THE TRIGONOMETRIC FUNCTION.
SOLVE FOR THE UNKNOWN
IS THERE AN UNKNOWN SIDE YET ?
YES
NO
WRITE ANSWERS
STOP

$\because m\angle B + m\angle A = 90°$
Why?
then $90° - 70° = 20°$
Why?
$m\angle A = 20°$

$a = ?$
$m\angle B = 70°$
$m(\overline{AC}) = 14$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$\tan 70° = \frac{14}{a}$
$= 2.7$
$a = \frac{14}{2.7}$
$a = 5.2$

$\sin 70° = \frac{14}{c}$
$.93 \simeq \frac{14}{c}$
$c \simeq \frac{14}{.93}$
$c \simeq 15$

$m\angle A = 20°$
$m(\overline{BC}) = 5.2$
$m(\overline{AC}) = 15$

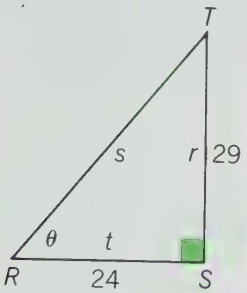
Problem 2

Given the measures of two sides of the right triangle find the measures of the two acute angles, and the third side.

Example 2

Given $\triangle RST$ with $m\angle S = 90°$, $m(\overline{RS}) = 24$, $m(\overline{ST}) = 29$, find $m\angle R$, $m\angle T$, $m(\overline{RT})$.

Solution



We could now use the Pythagorean Property to find $m(\overline{RT})$ but there is an easier way:

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan\theta = \frac{29}{24}$$

$$\tan\theta \simeq 1.2$$

$$\theta \simeq 50°$$

$$\sin 50° = \frac{29}{s}$$

$$.77 \simeq \frac{29}{s}$$

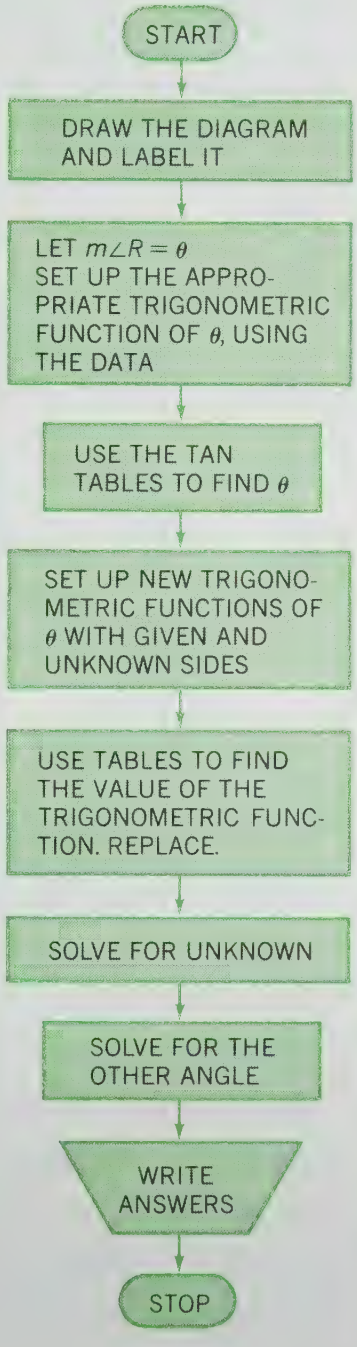
$$s \simeq \frac{29}{.77}$$

$$s \simeq 37.6$$

$$m\angle T = 90° - 50°$$

$$= 40°$$

$m\angle R = 50°$
$m\angle T = 40°$
$m(\overline{RT}) = 37.6$


START
DRAW THE DIAGRAM AND LABEL IT
LET $m\angle R = \theta$ SET UP THE APPROPRIATE TRIGONOMETRIC FUNCTION OF θ, USING THE DATA
USE THE TAN TABLES TO FIND θ
SET UP NEW TRIGONOMETRIC FUNCTIONS OF θ WITH GIVEN AND UNKNOWN SIDES
USE TABLES TO FIND THE VALUE OF THE TRIGONOMETRIC FUNCTION. REPLACE.
SOLVE FOR UNKNOWN
SOLVE FOR THE OTHER ANGLE
WRITE ANSWERS
STOP

Project

Project 3

Purpose

To determine a table of values of sine θ, cosine θ and tangent θ for $0° \le \theta \le 360°$.

To graph the values of sine θ, cosine θ and tangent θ for $0° \le \theta \le 360°$.

Method

1. Refer to Project 1.
2. Continue to draw radii at 10° intervals in each of the other 3 quadrants.
3. Label the coordinates of each radius at the circumference.
4. Record the data in a table similar to that of Table 2. To find tan θ use the method shown in Table 1.
5. Record the data from Table 2, Project 1 and the new data from this project on graphs as in Step 7 of Project 1.

Questions

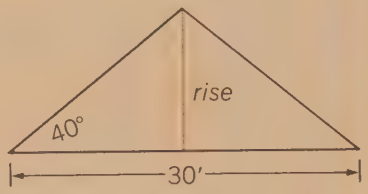
1. Describe the relation of the pattern of the sine θ graph and that of the cosine θ graph.
2. The tangent θ graph is very different from that of the other two. In what ways does it differ from sine θ? cosine θ?

Keep the results of this project for work in Chapter 13.

Use the data from Table 1 or Table 2 to answer the following questions.

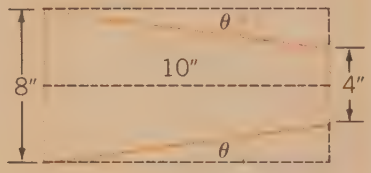
3. Interpolate in Table 2 to find:
   (a) sin 45° (b) cos 65° (c) tan 55°
   (d) sin 25° (e) cos 15° (f) tan 75°.
4. Read from the graph drawn in Project 1 to find the trigonometric functions in question 3. Compare the answers you obtained in the two methods. Which do you think are the more accurate: those from the graph or those obtained by interpolation from the table? Why?

5. A student standing 50 feet from the base of a tree finds the angle of inclination of the top of a tree to be 60°. Find the height h of the tree.



6. A carpenter wants the rafter of a house to form an angle of 40° with the plate. The width of the house is 30 feet. What is the rise of the rafter?

7. An engineer standing 250 feet from the base of a tower sights the top and finds the angle to be 60°. What is the height of the tower?



8. A machinist is turning down a shaft to the specifications shown in the diagram. Find the measure of the angle θ indicated.

EXERCISE 3·7

1. The cosine function decreases (in the first quadrant) as the angle increases. Explain in your own words what this means about the two sides of the triangle involved in the cosine relation.

2. The sine function increases (in the first quadrant) as the angle increases. Explain in your own words what this means about the two sides of the triangle involved.

3. Make a statement about the tangent function similar to those made above about the sine and cosine functions in questions 1 and 2.

3·8 Using Prepared Tables

In the previous sections you have established and used tables of values for the trigonometric functions. These tables are not very accurate. Most mathematics books contain tables that are correct to four decimals. Study this excerpt from a commercially prepared table of trigonometric functions.

Angle Degrees	Sine	Cosine	Tangent
0° 1°	.0000 .0175	1.0000 .9998	.0000 .0175
47° 48° 49°	.7314 .7431 .7547	.6820 .6691 .6561	1.0724 1.1106 1.1504
50° 51° 52° 53° 54°	.7660 .7771 .7880 .7986 .8090	.6428 .6293 .6157 .6018 .5878	1.1918 1.2349 1.2799 1.3270 1.3764
89°	.9998	.0175	57.2900
90°	1.0000	.0000	undefined

This table tells us sin 49° = .7547, cos 49° = .6561, and tan 49° = 1.1504, each correct to four decimal places.

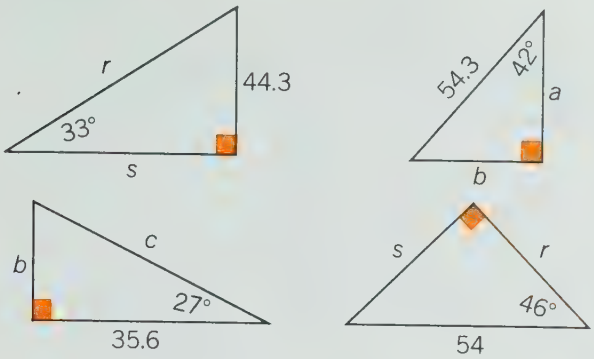
This table is for angles with measures 0° to 90° inclusive. In a later section you will learn how to work with angles whose measures are greater than 90°.

EXERCISE 3·8

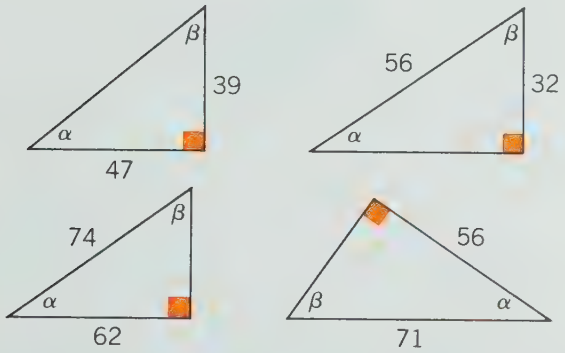
Use the table on page 264 to find the value of each.

1. sin 56°	2. tan 63°	3. cos 68°
4. tan 89°	5. cos 21°	6. sin 88°
7. sin 48°	8. cos 58°	9. tan 58°
10. sin 45°	11. cos 45°	12. tan 45°
13. sin 36°	14. cos 36°	15. tan 36°

16. Find the measures of the two unknown sides in each triangle.



17. Find
    (a) the measure of the two angles α, β.
    (b) the measure of the third side (by trigonometry).



18. Given: $\triangle AMP$, $m\angle A = 32°$, $m\angle M = 90°$, $m(\overline{MP}) = 63.2$; find $m\angle P$, $m(\overline{MA})$ and $m(\overline{AP})$.

19. Given: $\triangle RST$, $m\angle T = 90°$, $m(\overline{RS}) = 63$, $m(\overline{RT}) = 24$; find $m\angle R$, $m\angle S$, and $m(\overline{ST})$.

3·9 Some Applications of Trigonometry

Trigonometry may be used in many situations. It can be used in surveying, in engineering and in simplifying otherwise difficult problems. Its use in surveying will

be discussed briefly, while some of the other uses are illustrated in the exercises.

Surveyors use trigonometry to calculate

1. height of objects not readily measured by other methods;
2. angles between lines not measurable by other methods;
3. distances between objects not readily measurable by other methods.

Most of the cases to be studied can be represented by right triangles. Several terms need to be introduced to facilitate easy discussion.

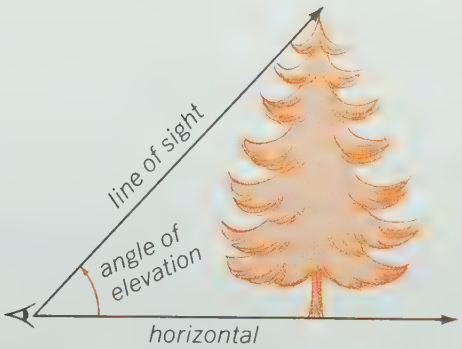


Figure 3-20

The line of sight is the line drawn from the observer's eye to the object being sighted.

The angle of elevation (inclination) is the angle determined by the line of sight of *some point above the observer* and the line representing the horizontal.

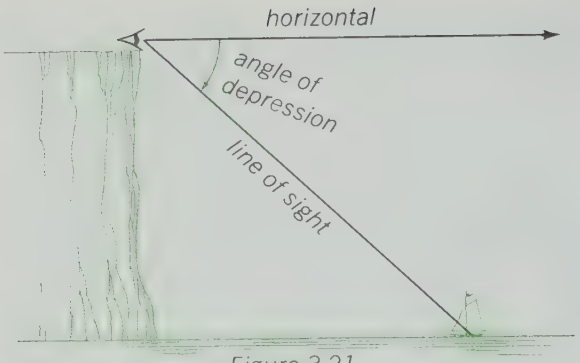
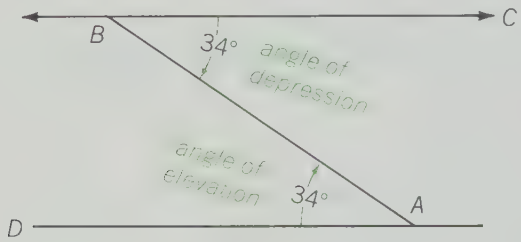


Figure 3-21

The angle of depression (declination) is the angle determined by the line of sight *of some point below the observer* and the line representing the horizontal.



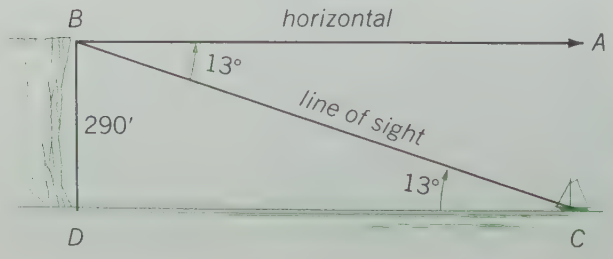
Note how the angle of depression of point *A* from point *B* gives the angle of elevation of point *B* from point *A*. Why?

This is based on the fact that *BC* is parallel to *AD*. *BA* is the transversal.

Alternate angles $\angle CBA$ and $\angle DAB$ are equal.

Example 1

An observer stands on a cliff overlooking the sea. The cliff rises 290 feet above the sea. The angle of depression of a sailboat at sea is 13°. Find the distance the sailboat is at sea. See Figure 3-21.



Solution

$$\because m\angle ABC = 13°$$
$$\therefore m\angle DCB = 13°$$

We want to find $m(\overline{DC})$. We know $m(\overline{DB})$.

$$\tan 13° = \frac{290}{m(\overline{DC})}$$
$$m(\overline{DC}) \tan 13° = 290$$
$$m(\overline{DC}) = \frac{290}{\tan 13°}$$
$$m(\overline{DC}) = \frac{290}{.2309} \qquad \text{(from tables)}$$
$$\simeq 1256 \qquad \text{(slide rule)}$$

The sailboat is 1256 feet out at sea.

EXERCISE 3·9

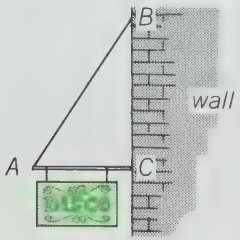
1. A student, using an inclinometer he made, recorded the angle of elevation of the top of his school to be 37° and the distance he was from the school to be 124 feet. What is the height of his school?

2. A firefighter plane flies over level terrain at an altitude of 5000 feet. An observer in the plane notes the angle of depression of a fire on the ground is 22°. Find the land distance the plane is from the fire.

3. A ladder 34 feet long leans against a wall. It makes an angle of 72° with the ground. How far up the wall does the ladder reach?

4. A boy flies a kite on a string 148 feet long. Assume the kite string is straight. The angle of inclination of the string is 63°. Find the height of the kite if the boy is 5 feet tall.

5. A sign is to be mounted on a wall as shown in the diagram. The outer point of the sign is 4 feet 5 inches from the wall. The company plans to use a supporting rod A to B such that B is 6 feet above C. How long must the rod be (do not allow anything for connections)?



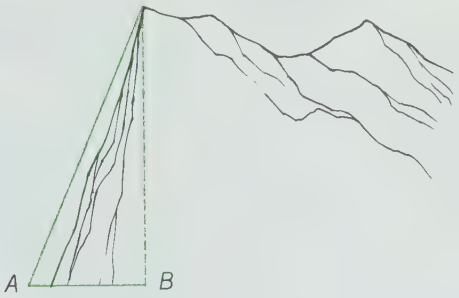
6. The height of a pylon is known to be 120 feet. It casts a shadow 85 feet long—measured from the centre of the base of the pylon. What is the angle of elevation of the sun?

7. A meteorological balloon 5 minutes after release radios its altitude as 1456 feet. Its angle of elevation from its release base is 22°.

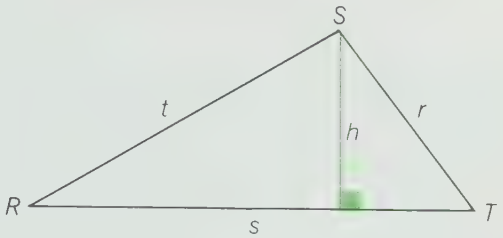
(a) Find the ground distance it has travelled since it was released.

(b) Find the ground speed in miles per hour of the balloon (equivalent to average wind speed).

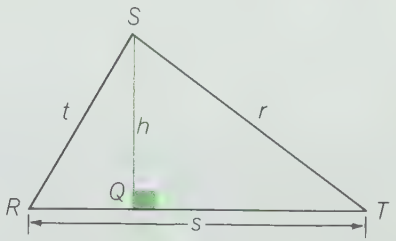
8. Find the height of the mountain given $m\angle A = 67°$, $m(\overline{AB}) = 3420$ feet.



9. The area of the triangle can be found given $r = 32$, $s = 68$, $m\angle T = 53°$. First find the measure of h. Then use the formula $A = \frac{1}{2}sh$. (Area is one half the base times the altitude.)



10. (a) Solve for h in $\triangle SQT$ given $m(\overline{ST})$ and $m\angle T = \theta$.



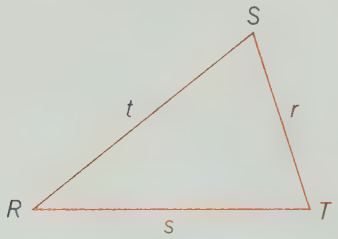
(b) Given area $= \frac{1}{2}sh$, replace h with the expression for h you obtained in part a.

(c) Use the new formula found in part b for the area of a triangle to find the area of $\triangle STR$ if $r = 19.2$, $s = 31.1$ and $\theta = 42$.

(d) Notice that in order to use this new formula you must be given the measures of two sides and the angle contained (formed by the two sides).

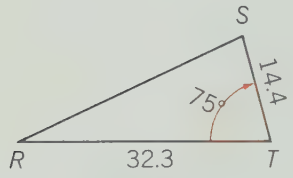
11. The area of $\triangle RST$ with acute $\angle T$ can be found by the formula

$$A = \tfrac{1}{2}rs \cdot \sin T.$$

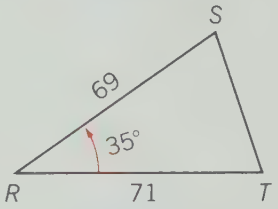


Find the area, given $r = 16.4$, $s = 24.3$ and $m^\circ\angle T = 72$.

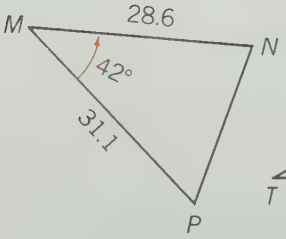
12. Find the area of each triangle where possible.



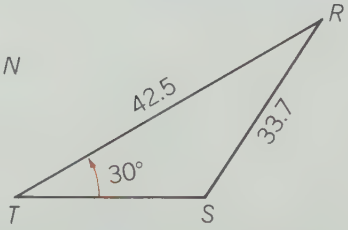
(a)



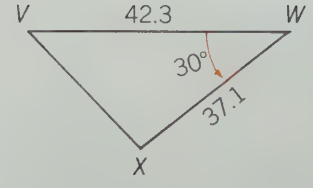
(b)



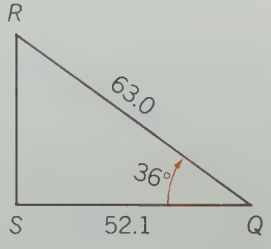
(c)



(d)

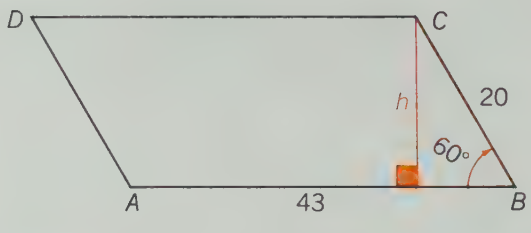


(e)



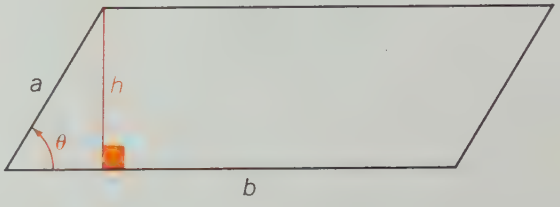
(f)

13. The area of a parallelogram is given by $A = bh$. Find h with the information shown. Then use $A = bh$ to find the area.



14. Repeat question 13 using $m(\overline{BC}) = 27$, $m(\overline{AB}) = 56$, and $m\angle ABC = 72°$.

15. In the figure shown below find an expression for h. In the expression $A = bh$ replace h with this expression. Then use the new formula for area of a parallelogram to find the area when $\theta = 52$, $b = 51$ and $a = 22$.



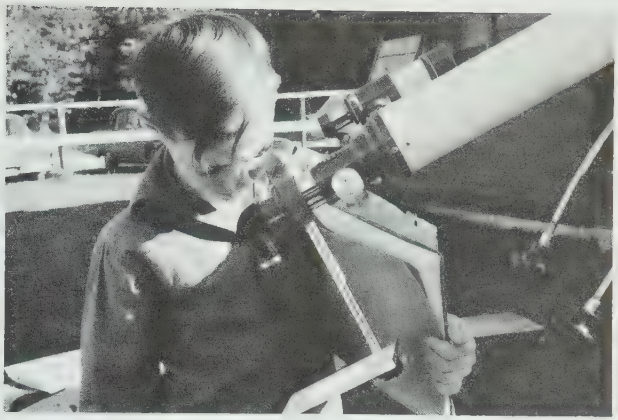
16. A ladder makes an angle of 67° with the level ground. How far should the foot of the ladder be from the wall of the house if the ladder is to reach a window ledge 26 feet from the ground?

17. A mountain roadbed has an inclination of 19° to the horizontal. What is the increase in altitude in a distance of 1450 feet measured along the road?

18. At 1:00 p.m. the angle of elevation of the sun is 53°. At 5:00 p.m. the angle of elevation is 26°. Find how much longer the 5 o'clock shadow of a tree 18 feet high is than the 1 o'clock shadow.

19. An isosceles triangle has a measure of 26 inches on each of its equal sides. The angle between the equal sides is 36°. Calculate the lengths of the three perpendiculars from the vertices to the opposite sides.

20. If you did not do the projects in Chapter 10 of Book 2, obtain a copy of Book 2. Make a selection from the projects in Chapter 10. Perform the projects.



Trigonometry is a particularly important subject for a budding astronomer. Why?

Using the Library

1. *The Giant Golden Book of Mathematics.* Read pages 56–58. Explain the similarities between the sine wave and the wave of the oscilloscope.

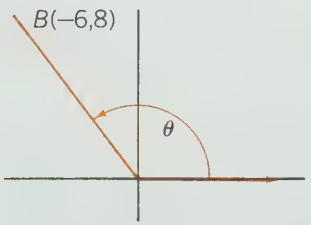
2. *Romping Through Mathematics.* Read pages 72 onward. Use the rule on page 83 to find the area of some triangle. Make up a problem to exchange with other classmates.

3. *Mathematics Enrichment.* Read pages 51–63.

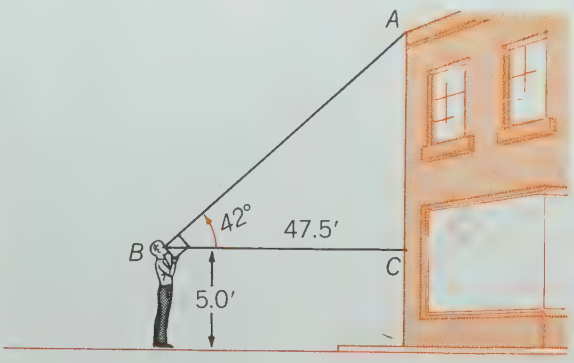
REVIEW EXERCISE 3

1. State in your own words what we mean when we say "Cosine 30° is 0.866."

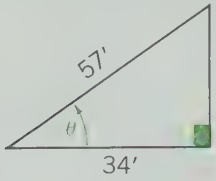
2. Find sin θ given the diagram below.



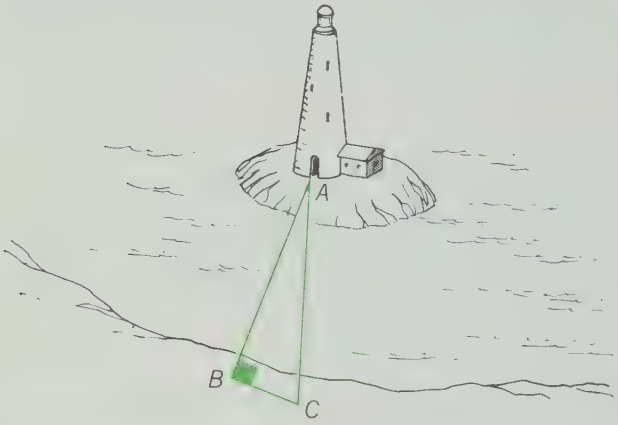
3. Using an inclinometer, a student surveyor measures the angle of inclination ($\angle ABC$) of a tall building as shown. From the information on the diagram find the height of the building.



4. Find the value of θ to the nearest degree.



5. A lighthouse stands on a rocky island some distance from shore. Bryan walks a line BC perpendicular to $\overleftrightarrow{AB}$. $\overline{BC}$ is 64 yards. Angle ACB is 72°. Find the distance AB.



We require a Service Garage to maintain our fleet. As each plant must account for their own fleet costs, it is necessary to operate a garage independently but controlled by one plant and that plant actually is selling mechanical repair service to the other plant. We compute monthly our actual labour cost summary to give us a "control" picture of how many actual hours of labour are sold so that the hourly labour rate charged to the other participating plant can be adjusted if necessary.

Work out actual cost per hour of garage labour and the actual efficient hours for the month's operation.

Payroll cost for two men – $764.40
Hours paid for on payroll – 336 hours

Labour sold to companies
Company – "A" – 171.4 hours @ 4.25 = $728.45
Company – "B" – 77.7 hours @ 4.25 = $330.23

How many hours were not productive?

What was actual cost per hour of garage labour?

When staff benefits are added to labour cost what is the actual cost per hour?

Summary of Benefits Share paid by company operating garage for two employees for the month.

A. Workmen's Compensation	– 8.53
B. Pension	– 13.45
C. Unemployment Insurance	– 8.66
D. Medical Insurance	– 23.25
E. Vacation Pay	– 4%

Chapter Four
commission

4·1 Salesmen's Commission

Often a business firm pays its salesmen a commission instead of a fixed salary. The commission is based on the value of sales or on the number of articles sold. This is called a *straight commission basis.*

To find the commission when the rate is based on the value of sales:

Example 1

Mr. Uptight, a car salesman, received 10% of his sales. During the month his sales totalled $14,000. What was the amount of his commission?

Solution

Commission = rate of commission × sales

Let commission be $\$x$.

Therefore
$$\$x = 10\% \times \$14{,}000$$
$$x = .1 \times 14{,}000$$
$$x = 1400$$

The commission is $1400.

To find the commission when the rate is based on the number of items sold:

Example 2

Mr. Bristle, the brush salesman, receives $.30 for each brush he sells. In one week he sold 200 brushes. What was his commission for selling brushes for the week?

Solution

Commission = rate of commission × number sold

Let commission be $\$x$.

$$\$x = \$.30 \times 200$$
$$x = .3 \times 200$$
$$x = 60$$

The commission is $60.00.

EXERCISE 4·1

Use slide rule or calculator whenever appropriate.

1. John Summer, a student, sold magazines and received a straight commission of 42% of his gross sales. In one month he sold $348 worth of magazines. What was his commission?

2. A part-time salesman receives $.55 for each book he sells. In one week he sold 98 books. How much did he earn in commission?

3. Find the commission in each case.

	Quantity sold	*Commission on each*
(a)	548	$6\frac{1}{2}$¢
(b)	496	$11\frac{3}{4}$¢
(c)	123	$1.26

4. Find the commission in each case.

	Sales	*Rate of commission*
(a)	$235	16%
(b)	$3,700	14%
(c)	$52,000	$4\frac{1}{2}$%

5. Which newsboy would make more money—the boy who receives $.03 commission on each of the 125 newspapers he delivers or the boy on the corner who gets 20% of the sale of 186 newspapers sold at $.10 each?

4·2 Finding Sales When Rate and Commission are Known

Example 1

A hardware salesman reports he made $20,000 in a year selling hardware which pays a commission of 8% of sales. What were the salesman's gross sales?

Solution

Commission = rate of commission × sales

Let gross sales be $\$x$.

$$\$20{,}000 = 8\% \times \$x$$
$$20000 = 0.8x$$
$$x = \frac{20000}{.08} \qquad (3)$$
$$x = 250{,}000$$

The gross sales were $250,000.

From (3) we are able to see

$$\text{sales} = \frac{\text{commission}}{\text{commission rate}}$$

EXERCISE 4·2

1. A paint salesman worked on a commission rate of 22%. His commission one month was $814. What was the amount of his sales?

2. An employee of a soft water company receives $.75 for each refill tank he installs. He received $350 in one month. How many refill tanks did he install?

3. Find the sales, or the number sold, in each case.

	Commission Rate	Commission	Sales or Number sold
(a)	4%	$364.00	_____
(b)	17%	$496.40	_____
(c)	49¢ each	$3,177.00	_____
(d)	$1.56 each	$171.60	_____

4·3 Graduated Scale

Companies sometimes offer their salesmen a graduated commission plan. This plan pays one commission rate for the initial sales then another rate for subsequent sales.

Example 1

A salesman is paid 5% commission on the first $5,000 of monthly sales and 8% on all sales over $5,000 (called excess sales). His monthly sales totalled $9,400. What was his commission?

Solution

Step 1. Find excess sales: $9400 - 5000 = \$4400$

To calculate commission:

Step 2. Find commission on first $5000:

$.05 \times 5000 = \$250$

Step 3. Find commission on excess:

$.08 \times 4400 = \$352$

Step 4. Find total commission: $602

The salesman's commission was $602.

EXERCISE 4·3

1. Roger, a real estate salesman, receives a commission on a graduated plan: 8% on the first $100,000 of sales in a year, then 10% on the excess over $100,000. In a year Roger sold $145,600 worth of real estate. What was his commission?

2. A salesman for "Know-It-All" encyclopedia receives \$45 for each of the first 100 sets he sells, then \$65 for each set in excess of 100 up to and including 200, and then \$85 for each in excess of 200. Find his commission if he sells 265 sets.

3. Find the commission in each case:

	Total sales	*Commission rates*
(a)	\$450	4% on first \$200, 5% on excess over \$200
(b)	\$2,860	2% on first \$1500, 1% on excess over \$1500
(c)	4000 subscriptions	\$1 on each of first 2000 subscriptions, \$1.25 on excess over 2000.

4. The *Brush Less Brush Company* pays its salesmen weekly on the basis of Plan A or Plan B.

   Plan A – 40% on the first \$400 of sales, then 35% of all sales over the \$400.

   Plan B – 50% of the first \$300 of sales, then 30% of all sales over \$300.

   (a) Complete Table 1.

Table 1

Rate	Dollar sales							
	100	200	300	400	500	600	700	900
Plan A								
Plan B								

   (b) Graph the salaries possible under the two plans (Table 1) with reference to one set of axes.

   (c) Do you think a salesman could earn more under Plan A than under Plan B? If so, when?

5. Why would a company rather pay salesmen a commission than a fixed salary?

6. Why would a company pay salesmen on a graduated commission plan?

4·4 Salary Plus Commission

Some companies, especially stores, pay their salesmen a fixed salary plus a commission on the sales. The commission plan varies from company to company. The fixed salary is called the *drawing account* while the commission plan is called the *quota.*

Example 1

A hardware store pays its salesman a fixed salary of \$350 a month plus 2% of all sales. What are the salesman's total earnings if his monthly sales amounted to \$1246?

Solution

1.	Drawing account	\$350.00
2.	Quota .02 × 1246	24.92
3.	Total earnings	\$374.92

total salary = drawing account + quota

EXERCISE 4·4

1. Find the total earnings of each salesman:

Salesman	Monthly sales	Drawing account	Quota
A	\$ 993	\$413	$4\frac{1}{2}$% of sales
B	1,450	258	$5\frac{3}{4}$% of sales in excess of \$600
C	2,296	212	$6\frac{1}{4}$% of sales in excess of \$1,500 up to \$2,000 $8\frac{3}{8}$% of sales in excess of \$2,000
D	2,606	342	10% of all sales plus bonus of \$50 if sales total more than \$2,500

2. Jane Richards, a saleslady in the Valiant Department Store, is paid $85 a week plus 2% on sales in excess of $900. In one week her sales amounted to $1356. Find her total earnings for the week.

3. In one week Mary Jones earned $86 on a salary of $50 a week plus $2\frac{1}{2}\%$ of sales commission. What were her weekly sales?

4. Working on a weekly salary of $40 plus a commission of $7\frac{1}{4}\%$ of sales, Dick Harrigan earned $98. What were his weekly sales?

5. You live in a small city of 100,000 and work in a clothing store. You are offered a choice of three commission plans.

   I Straight Commission: 15% of weekly sales

   II Graduated Commission: 10% of the first $400 of sales plus 20% of excess of $400 (weekly)

   III Salary Plus: $30 weekly plus 5% of sales weekly

   (a) Which would you choose? Why? Discuss.

   (b) Calculate the total earnings under each plan if your weekly sales were:
   (i) $100 (ii) $400
   (iii) $700 (iv) $900.

4·5 Finding Rate

Example 1

A salesman, on a straight commission, had sales of $150,000. His commission for that period of time was $4,800. What was the rate of commission?

Solution

Let rate of commission be $x\%$.

commission = rate of commission × sales

$$4800 = \frac{x}{100} \times 150{,}000$$

$$4800 = x \times 1500$$

$$x = \tfrac{4800}{1500} = 3.2.$$

The rate is 3.2%.

If the earning plan is one of a *salary plus*, the solution is in two parts.

Example 2

Miss Nova receives a drawing account of $35 a week plus a percentage of sales. In one week her sales are $340. Her earnings are $47.90. What is the rate paid on sales?

Solution

The amount of her earnings from the commission is:

Total salary – drawing account = quota

$47.90 – $35 = $12.90

Let rate be $x\%$.

commission = rate of commission × sales

$$12.90 = \frac{x}{100} \times 340$$

$$x = 3.8.$$

The commission rate is 3.8%.

$C = R \times S$

EXERCISE 4·5

1. Find the rate of straight commission when Mr. Bryan, a salesman, earns $841.50 on $12,460.

2. Janet is a salesgirl in a bridal shop. Her drawing account is $45 a week. Her sales for the week are $420. She receives a commission on her total sales. Her total earnings for the week were $69.28. What is the rate of commission?

3. Find the rate of commission in each case:

	EARNINGS	SALES	PLAN
(a)	$98.56	$896	straight commission
(b)	$87.39	$456.50	$60 plus ■% of sales
(c)	$93.52	$860	$70 plus ■% of sales over $200

4·6 Graphical Illustrations

In Exercise 4·4, question 5, you were asked which plan was the best:

I 15% of all sales

II 10% of the first $400 of sales plus 20% of the excess of $400 (weekly)

III $30 weekly plus 5% of sales.

You then possibly noticed that if the sales were $100, plan *III* produced the highest total earnings. (Hence, the best!) If the sales were $700, plan *I* produced the highest total earnings, while if the sales were $900, plan *II* produced the highest total earnings.

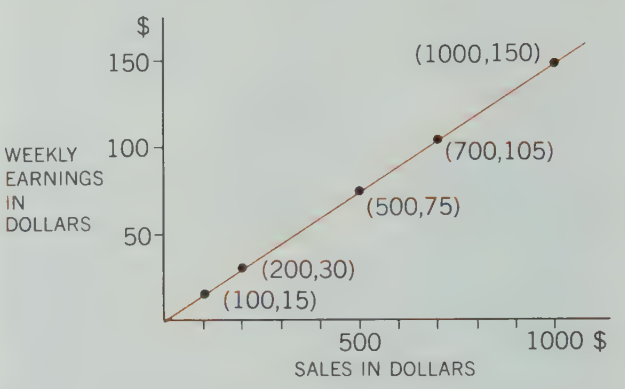
When do the total earnings of
(a) plan *I* equal those of plan *II*?
(b) plan *I* equal those of plan *III*?
(c) plan *II* equal those of plan *III*?

How shall we go about answering these three questions? Shall we guess, then check our guess? Perhaps there is a better way. Let us use graphs.

Let us graph the relation of the earnings to the sales of plan *I*. We will make a table first.

SALES $	0	100	200	500	700	1000
EARNINGS $	0	15	30	75	105	150

We now record the data on a graph and draw a smooth curve to contain the points.



(a) Is this a direct variation? Why do you think so?

(b) Is this a linear function? Why do you think so?

(c) If this is a linear function and a direct variation it should fit the general form of $y = mx$. What is the y in this case? m? x?

We have shown you how to graph the earnings of the first plan. Now graph, on the same graph as plan *I*, the earnings of plan *II* in a similar way. In this case, since the commission rates change, care must be exercised in choosing values of the variable *sales*. Choose, among others, the changing point and one other slightly above it. In this case, choose $400 and $405 as two values of *sales*.

SALES $	0	···	400	405	500
EARNINGS $					

Complete the graph with a color of pencil different from that used for plan *I*.

Now graph on the same graph as plans *I* and *II* the earnings of plan *III*. Use a third color for the graph. Label the three graphs *Plan I*, *Plan II*, *Plan III*, respectively.

SALES $	0	100	200	
SALARY $				

(d) Is the graph of plan *II* portions of two straight lines? Is that portion of the graph for $0 ≤ sales ≤ $400 a graph of a direct variation? of a partial variation? of a linear function? Why do you think so?

(e) Is the graph of plan *III* a graph of a direct variation? of a partial variation? of a linear function? Why do you think so?

(f) The algebraic form of the relation of plan *III* should fit the general form of $y = mx + b$. Why? What quantity does y stand for? does m stand for? does b stand for?

Now that we have the three graphs drawn let us try to answer our three questions.

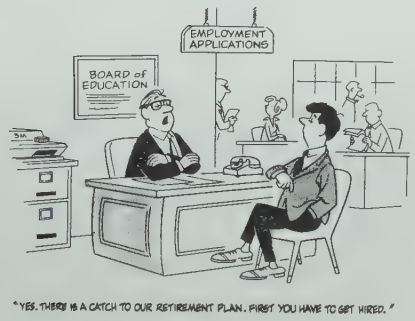
(a) When do the total earnings of plan *I* equal those of plan *II*? Study the graphs of plans *I* and *II*. Do the two lines intersect? What are the coordinates of the point at which they intersect? What is the amount of sales and of earnings at the time the two plans are equal?

(b) Answer the other two questions on page 66.

(c) Would a graph like this help you to decide which plan you would accept (as an employee)?

(d) What other information would you like to have before making a decision as to which plan you would choose?

(e) How would you answer the employer if you were applying for this job and he made this offer? Would you like to work two weeks before you make up your mind? Why?



"YES, THERE IS A CATCH TO OUR RETIREMENT PLAN. FIRST YOU HAVE TO GET HIRED."

EXERCISE 4·6

1. Two salesgirls were working at the same job. On the average their sales were the same. Jill Neam had a drawing account of $55 a week with a quota of 3% of sales. Lily Mean had a straight commission of 12% of sales. Draw suitable graphs of, and tell what you can about the earnings of the two girls.

2. A department store offers its sales personnel two weekly plans:

   I $30 drawing account with a quota of 20% of all sales over $500

   II 10% commission on first $400 of sales then 15% of all sales in excess of $400.

   Draw suitable graphs of the earnings under the two plans. Tell what you can about the earnings under the two plans.

3. A firm offers you, a new salesman, three plans of payment:

   I 20% of the first $150 of sales, 40% on excess

   II $25 drawing account, plus 10% on first $200 sales then 50% on excess

   III $50 drawing account, plus 20% on first $100 sales.

   (a) Draw graphs (on one set of axes) to show the earnings under the three plans. Label each.

   (b) Tell what you, as an employee, would be concerned about with regards to the three plans.

   (c) Under what conditions would you select
   (i) plan *I*? (ii) plan *II*? (iii) plan *III*?

4. Marylynne sells cosmetics on a straight commission basis. In one month her sales totalled $290. Her commission was $130.50. Draw a graph to represent the relation between earnings and sales. Determine from the graph the rate of commission.

   To help you:

   (a) What were her earnings if her sales were $80?

   (b) Are the earnings on a straight commission a linear function of the sales?

   (c) Is the graph a straight line from the origin through the point (290, 130.50)? Why?

   (d) Rate of commission (percent) is the commission for $100 of sales. Locate on the straight line the point whose first coordinate is 100. What is the second coordinate?

   (e) What is the rate of commission?

   (f) Check by using a formula.

5. Buck Thorne sells motorcycles for the Free Wheelers Company on a straight commission. His sales one month are $3500 and his commission is $525. Draw a graph to show the relation between commission and sales. From the graph determine the rate of commission. Check by using a formula.

4·7 Commission to Agents

It is not always convenient or desirable for a businessman on behalf of himself or his company to do his own buying and selling. In such cases he employs someone to do the buying and selling for him. One who is employed to buy or sell for someone else is known as a *commission agent* or a *commission merchant.* The person for whom the agent acts is known as the *principal.* The agent charges for his services. He may charge a certain percentage of the sales price.

Example 1

(Based on sales) A fur trapper, Mr. Wright, decides to sell his trapline and retire in Paradise City. ABC Realty Company agrees to sell Mr. Wright's line for 12% of sale price. The sale price is $28,000. What is ABC Realty Company's sales commission? How much did Mr. Wright realize from the sale?

Solution

Commission = rate of commission × sales

Let $\$x$ be the sales commission.

$$\$x = 12\% \times \$28{,}000$$
$$x = .12 \times 28000$$
$$x = 3360.$$

The commission is $3,360.

Mr. Wright realized

$$28{,}000 - 3{,}360 = \$24{,}640$$

sale price − commission = net proceeds

Example 2

(Based on purchases) A lumber yard, The Plank Company, employs an agent to buy 100,000 (100M) board feet of lumber. The agent charges $4\frac{1}{2}\%$ of sales. The agent managed to get the required stock at $87 a thousand board feet. How much was the agent's commission, and how much did The Plank Company pay for the lumber?

Solution

$87 buys 1000 board feet. The agent buys 100,000 board feet. Therefore, the sales are $100 \times 87 = \$8700$.

Now we will calculate the commission:

$$\text{commission} = \text{rate} \times \text{sales (purchases)}$$

Let commission be $\$x$.

$$x = .045 \times 8700$$
$$x = 391.50$$

The agent's commission was $391.50.

To calculate the total cost to The Plank Company we use

$$\text{total cost} = \text{purchase price} + \text{commission}$$

$$= 8700 + 391.50$$
$$= 9091.50$$

Total cost was $9,091.50

Did you notice the two examples are different? In the first example Mr. Wright employed the agent to sell something for him. In this case, the agent's commission was subtracted from the sale price. In effect, the agent paid Mr. Wright $28,000 and Mr. Wright paid the agent $3,360 leaving Mr. Wright with a balance of $24,640 – the net proceeds.

In the second example, the agent's commission was added to the sale price. In this instance The Plank Company paid $8,700 for the lumber *and* paid the agent $391.50 for handling the sales. Hence, the total cost to The Plank Company was $9,091.50.

In a similar way analyse carefully each situation in the following exercises.

EXERCISE 4·7

1. The Cattlemen's Feeders Association employs an agent to buy 50,000 bushels of oats at $1.22 a bushel. It is agreed the agent will receive $2\frac{1}{4}\%$ commission.
   (a) Find the sale price of the oats.
   (b) Find the agent's commission.
   (c) Find the total cost of the oats to the principal.

2. The same Cattlemen's Feeders Association employed another agent to sell 350 head of feeder stock. The sale price for the stock was $82,250. The agent's commission was $1\frac{1}{8}\%$ of sales.
   (a) Find the agent's commission.
   (b) Find the net proceeds to the Association.
   (c) Construct a flow chart for part (b).

3. The British Columbia Fruit Growers Association commissions an agent to obtain for them, delivered to Kelowna, 2000 tons of a particular fertilizer. The commission rate was $\frac{1}{2}$ of 1 percent (.005) of sales price. The agent obtained the fertilizer in Edmonton for $195,000. The expenses of delivering it to Kelowna were $3,250. What was the total cost of the fertilizer to the Association?

4. A commission agent, on behalf of his principal, sold $52,500 worth of property at a commission rate of $2\frac{1}{4}\%$. The expenses incurred were $222.80. What were the net proceeds to the principal?

5. Complete the chart.

Type of transaction	Sale price	Commission rate	Commission amount	Agent's expenses	Net proceeds or total cost
selling	$ 3,250	$8\frac{1}{4}\%$		$156	
selling	$ 5,418		$811.70	none	
buying	$15,450	$6\frac{1}{8}\%$		$670	
selling		15%		$240	$3,240
buying		15%		$240	$3,240

6. Why might one not wish to do his own buying and selling – as a private individual? as an employee of a firm?

Using the Library

1. *The Giant Golden Book of Mathematics.* Read pages 69-70.
   Why must a mouse produce more heat in cold weather for its volume than a polar bear?
   Why are animals in the Arctic usually large?
2. *Mathematics Enrichment.* Read pages 19 to 50 for interesting facts in the history of numbers.

REVIEW EXERCISE 4

1. Brian agrees to sell chickens for a straight commission of 20% of sales on the first $200 of monthly sales, then 30% of all monthly sales over $200. His sales in October were $440. What was his commission on the chickens?

2. In one month Joanne earned $480 on a salary of $360 a month plus 3% of sales commission. What were her monthly sales?

3. A company offers its new salesmen three employment plans.

   I 18% of the first $200 of sales, 45% on excess.

   II $50 drawing account, plus 8% on the first $200 sales, then $50 on excess.

   III $100 drawing account, plus 20% on the first $200 sales.

   (a) Draw graphs (on one set of axes) to show the earnings under the three plans.

   (b) Tell what you can about the three plans.

4. An agent, on behalf of his principal, sold $32,000 worth of property on a commission of $3\frac{1}{4}\%$. The expenses incurred amounted to $154.40. What were the net proceeds for the principal?

5. A Realty Agency sold a house at a 12% commission and received a commission of $864. What was the sale price of the house? What were the net proceeds the principal received for the house?

We are in the Commercial and Domestic Laundry, Dry Cleaning, Linen Rental and Dust Control Field.

A route salesman selling and servicing a dry cleaning and laundry home delivery route is paid by commission on sales handled with a weekly draw.

Sales volume is totalled and applied to classification volume relative to percentage of commission paid for each classification and his earnings balanced after the end of each month.

Summary of sales, percentage of commissions and draw already paid for typical month.

Laundry Sales: $800.00
Less credits and cancellations: $41.00
Comm. paid @ 15% =

Dry Cleaning Sales: $1,500.00
Less Agent Business: $200.00
Less credits and cancellations: $23.00
Comm. payable @ 17% =

Rental Business: $400.00 @ 10% =

Agency Business (as above): $200.00 @ 10% =

Specialty Sales – rugs, etc.: $500.00 @ 10% =

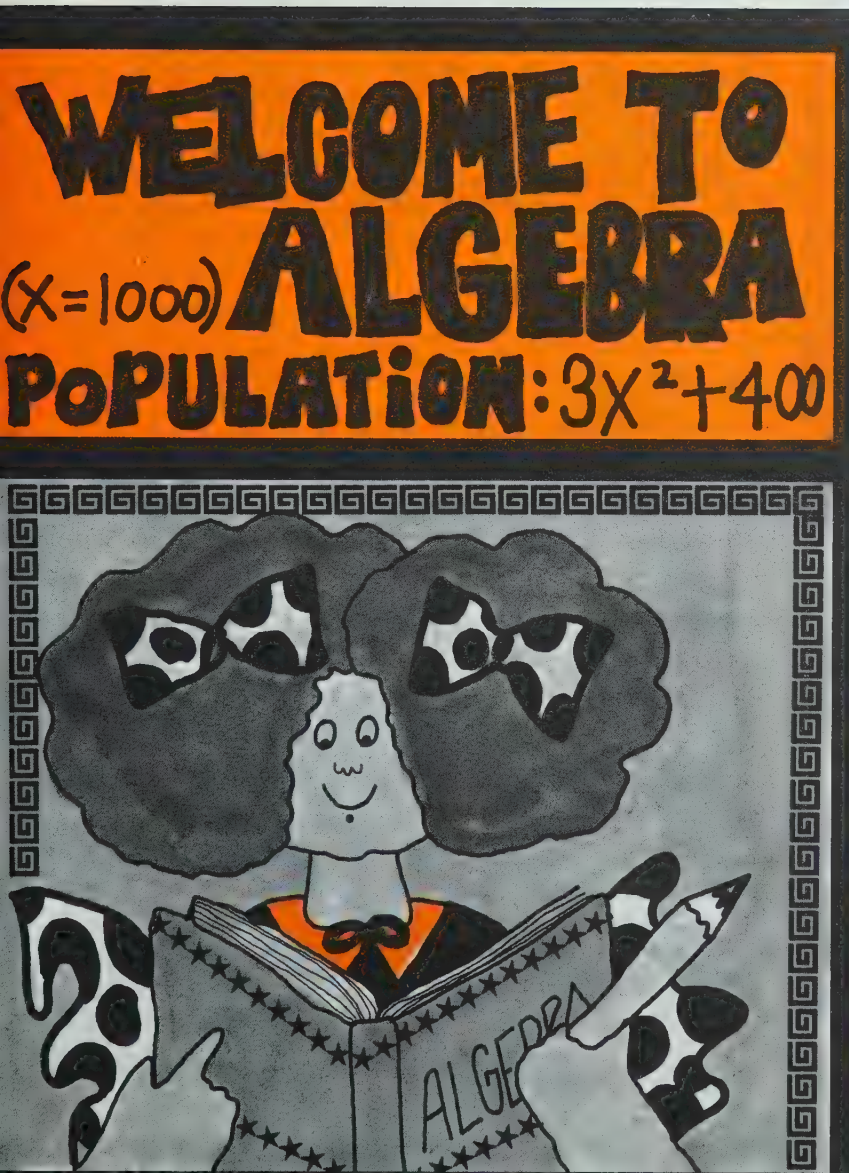
Total Commission Payable =

Pay drawn in lieu of commissions for month
Allowance: $80.00 for 44 hours
Has drawn pay for 21 days on an 8 hour day basis.
Total Drawn =
Total Comm. earned =
Less Draw =

Balance payable or deductable from next cheque =

What is delivery and sales labour cost as a percentage of total volume handled by route?

Chapter Five
ALGEBRA REVISITED



5·1 Decimals All Around Us

In the history of mankind frequent references are made to *pi*—that number which expresses the ratio of the circumference of a circle to the diameter. The Rhind papyrus gave $\pi = (\frac{4}{3})^4$. The Bible, in *I King's 7*, describes Solomon's Temple as having had a circular basin of "10 cubits from one brim to the other" and a "line of thirty cubits did compass it round about". Archimedes, second century B.C., found $3\frac{10}{71} < \pi < 3\frac{10}{70}$. Ptolemy, about 150 A.D., set $\pi = \frac{377}{120}$. The early Chinese set $\pi = \sqrt{10}$ and later, about 530 A.D., refined the value to $\pi = 3.1416$.

Archimedes figuring out a math problem.

Since the advent of computers the value of *pi* has been calculated to many decimal places. In 1967 the value of *pi* to 500,000 decimal places was determined by an electronic computer. This value starts

$$\pi = 3.141592653589\ldots.$$

In the series of questions that follow, you will learn more about irrational numbers. You will also make use of certain properties of the set of rational numbers and you may realize certain characteristics of the set of irrational numbers. Summarized, here are some general statements about the two sets.

The set of real numbers R is the union of the set of rational numbers Q and the set of irrational numbers $\overline{Q}$.

Each rational number can be expressed by an equivalent decimal fraction. These decimal fractions are called *periodic* in that part or all of the decimal repeats. If the part that repeats is zero, we call this decimal *terminating*, i.e.,

$$\tfrac{3}{4} = 0.7500\ldots \qquad \tfrac{1}{3} = 0.333\ldots$$

In the illustration 0.75 is a terminating decimal. If the portion that repeats is not zero it is a *repeating* decimal. In the illustration, 0.333 ... is a repeating decimal. This decimal is commonly written $0.\dot{3}$. The dot over the 3 indicates that the 3 repeats. The decimal 0.123123123... is written $0.\dot{1}2\dot{3}$. The dots over the 1 and 3 indicate the block of digits 123 repeats in that order.

Irrational numbers cannot be expressed in the form $\frac{a}{b}$ where a and b are integers.

Irrational numbers cannot be written as periodic decimals but rather are expressed as nonperiodic decimals.

Decimal approximations are written for both nonterminating and nonperiodic decimals. The rules for rounding off decimals are illustrated here:

$0.7432114 \simeq 0.743$
$0.7436461 \simeq 0.744$
$0.743512 \simeq 0.744$
$0.7435 \simeq 0.744$
$0.7445 \simeq 0.744$

} These 3 give difficulty. Study them carefully.

For further illustrations of the three types that give difficulty observe these:

$0.7448 \simeq 0.74$		$0.7348 \simeq 0.73$
$0.7449 \simeq 0.74$		$0.7349 \simeq 0.73$
$0.7450 \simeq 0.74$	← *NOTE!* →	$0.7350 \simeq 0.74$
$0.7451 \simeq 0.75$		$0.7351 \simeq 0.74$
$0.7452 \simeq 0.75$		$0.7352 \simeq 0.74$
$0.7460 \simeq 0.75$		$0.7360 \simeq 0.74$

When the digit 5 (or the digit 5 followed by one or more zeros) is dropped in rounding off a number, the preceding digit is increased by one if it is odd but left unchanged if it is even.

Notice that this rule is a refinement of the rule you have used previously. This is the rule most often used in science and more rigorous mathematics. While 0.75 is an approximation for 0.746 we say 0.75 is correct to two significant figures so we use the *equal* sign: $0.746 = 0.75$.

EXERCISE 5·1

1. What is the approximation for π implied by *I King's 7* in the Bible? Express as a decimal numeral, correct to 4 decimals the value of π from the Rhind papyrus.

2. (a) Express as decimal numerals to 4 decimal places, the two values between which the value of π occurs according to Archimedes.

   (b) What is the average to 4 decimal places of the two values found in (a)?

   (c) Find the average of $3\frac{10}{70}$ and $3\frac{10}{71}$ in fraction form. Then express this in decimal form correct to 4 decimal places.

   (d) Are the two averages in (b) and (c) the same? Do you have a comment?

   (e) How much does the value of π in (b) differ from the computer value rounded to 4 decimal places?

   (f) What is the percentage error of the value in (b)? Take the computer value rounded to 4 decimal places as the correct value:

   $$\text{percentage error} = \frac{\text{error}}{\text{correct value}} \times 100.$$

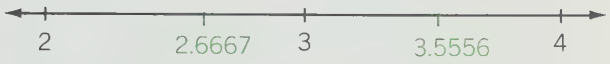
   (g) Find, by Newton's method, the approximate value of $\sqrt{10}$ correct to 4 decimal places.

   (h) Use your value of $\sqrt{10}$ and compare this value for π to the computer value, correct to 4 decimal places.

   (i) Find the percentage error of the Chinese value of π compared to the computer value (each correct to 4 decimal places).

3. Work this set of questions. Express each answer as a decimal correct to 4 places. Then show each number on the number line. The first two parts are done for you.

   (a) $2 \times \frac{2}{1} \times \frac{2}{3} = \frac{8}{3} \simeq 2.6667$
   (b) $2 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} = \frac{32}{9} \simeq 3.5556$



   (c) $2 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} = \blacksquare \simeq \blacktriangle$
   (d) $2 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} = \blacksquare \simeq \blacktriangle$
   (e) $2 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} = \blacksquare \simeq \blacktriangle$

   This is one way the value of π has been found. If you continue to extend the series and find the product in each case, you will see that the product gets closer and closer to π. If you have a calculator find the product of 10 terms of this series. Compare this value to the computer value of π.

4. Find the sum for this series and map the sums on the number line as shown. (Express each as a decimal correct to 4 decimal places.)

   (a) $4 - \frac{4}{3} = \blacksquare \simeq \blacktriangle$
   (b) $4 - \frac{4}{3} + \frac{4}{5} = \blacksquare \simeq \blacktriangle$
   (c) $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} = \blacksquare \simeq \blacktriangle$
   (d) $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} = \blacksquare \simeq \blacktriangle$
   (e) $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} = \blacksquare \simeq \blacktriangle$

   (i) What number does the above series approach?

   (ii) Compare the answer in (e) to the correct value rounded to 4 decimal places.

5. A very difficult series is the continued fraction series here. Record each approximation on the number line.

(a) $\dfrac{4}{1+1^2}$

(b) $\dfrac{4}{1+\frac{1^2}{2}}$

(c) $\dfrac{4}{1+\dfrac{1^2}{2+3^2}}$ Work the portion in the dotted ring first.

(d) $\dfrac{4}{1+\dfrac{1^2}{2+\frac{3^2}{2}}}$ Work from the inner circle out in each succeeding case.

(e) $\dfrac{4}{1+\dfrac{1^2}{2+\dfrac{3^2}{2+5^2}}}$

(f) $\dfrac{4}{1+\dfrac{1^2}{2+\dfrac{3^2}{2+\dfrac{5^2}{2+7^2}}}}$

What number do you think this continued fraction approaches?

6. Continued fractions are used in a number of places. The following continued fraction approximates $\sqrt{8}$. Find the value of each as you have done in question 5.

(a) $2+\dfrac{1}{1+1}$

(b) $2+\dfrac{1}{1+\dfrac{1}{4+\dfrac{1}{1+1}}}$

(c) $2+\dfrac{1}{1+\dfrac{1}{4+\dfrac{1}{1+\dfrac{1}{4+1}}}}$

Calculate by the method of successive approximations the value of $\sqrt{8}$ to 4 decimal places. Compare this value to the value you obtained for the continued fraction in c above.

Round each of the following to (a) 3 significant figures, (b) 2 significant figures.

7. 0.4316 8. 0.4345 9. 0.43455 10. 0.4351

11. 0.4445 12. 0.44455 13. 0.4450 14. 0.9755

Project

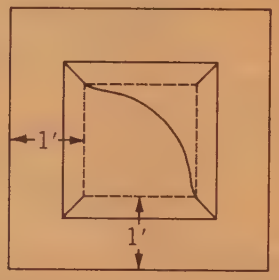
Project 1

Problem

To find experimentally an approximation for π.

Method

1. Obtain a sheet of cardboard about 16 inches square.
2. Cut and fold up edges as shown to produce a box.



3. With one corner as centre, and radius the width of the box, describe an arc as shown.
4. Place the box on the floor and mark a line 1 foot from each side of the box.
5. Standing with toes just touching a line on one side, toss 50 beans (or some other suitable replacement – pennies) into the box.

6. Record the number of beans that fall within the quarter circle drawn in the box.
7. Record the number of beans that fall in the box.
8. Repeat steps 5, 6, and 7 for the other three sides of the box.
9. Record the information in the chart:

TOSS	NUMBER OF BEANS IN SQUARE	NUMBER OF BEANS IN QUARTER CIRCLE
1 2 3 4		
TOTAL	S	C

Information

To calculate π, we use the following reasoning:

Let the length of the box be x.

The area of the quarter circle is $\pi \frac{x^2}{4}$. { Why divide by 4?

The area of the box is x^2.

The ratio of the area of the quarter circle to the area of the square is:

$$\frac{\text{area of quarter circle}}{\text{area of square}} = \frac{\frac{\pi x^2}{4}}{x^2} = \frac{\pi}{4}$$

The ratio of the number of beans in the quarter circle to the number of beans in the box is:

$$\frac{\text{ratio of beans in circle}}{\text{ratio of beans in box}} = \frac{C}{S}$$

These two ratios should be the same, hence

$$\frac{\pi}{4} = \frac{C}{S}$$

Therefore, $$\pi = \frac{4C}{S}$$

10. Use your value of C and S to find π.

11. One member from each group in your class writes the values of C and S on the chalkboard.

GROUP	C	S
1 2 3 4		
TOTAL		

12. Use these total values for C and S to find the experimental approximation for π to 5 decimal places.
13. Compare your approximation of π with the approximation 3.14159. How do they compare? How many ways can you make a comparison?
14. Which approximation of π, your group value or the class value, is the closest to the given value? What reason can you give for this?
15. Use as the true value of π, the approximation 3.14159. Use the value from Part 12 as calculated value. What is the percentage error of your calculation?

5·2 Working With Formulas

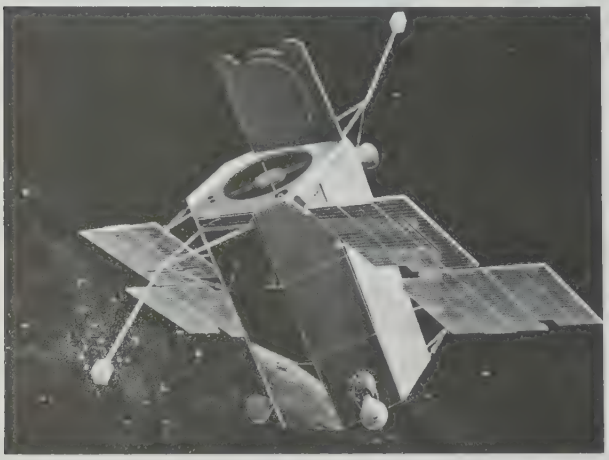
Frequently in a problem one is given a formula and certain data, then asked to solve for one of the variables. Basically, the problem is one of making certain replacements then solving for the unknown. This is a practical side to mathematics.

Example

The velocity v in miles per hour of a satellite orbiting the earth at an altitude of h miles in t hours, is given by

$$v = \frac{2\pi(3960 + h)}{t}$$

Find the height of a satellite which makes one revolution every 24 hours. Its average velocity is 6800 miles per hour. (Use $\pi = 3.14$.)



Solution

Method 1. We make the appropriate replacements in the formula.

$$v = \frac{2\pi(3960 + h)}{t}$$

$$6880 = \frac{2 \times 3.14\,(3960 + h)}{24}$$

$$6880 \times 24 = 6.28\,(3960 + h) \quad \text{(mult. by 24)}$$

$$\frac{6880 \times 24}{6.28} = 3960 + h$$

$$\frac{6880 \times 24}{6.28} - 3960 = h$$

Therefore, $h = 22{,}300$

Method 2. Solve for h first.

$$v = \frac{2\pi(3960 + h)}{t}$$

$$vt = 2\pi(3960 + h) \quad \text{(mult. by } t\text{)}$$

$$\frac{vt}{2\pi} - 3960 = h \quad \text{(div. by } 2\pi \text{ and subt. 3960)}$$

Now make the appropriate replacements

$$\frac{6880 \times 24}{2 \times 3.14} - 3960 = h$$

$$h = 22{,}300$$

The height is 22,300 miles

EXERCISE 5·2

Solve each of the following for the indicated variable. Several have been started for you.

1. $v = lwh$ (w)

2. $A = \pi r^2$ (r)

3. $F = \frac{9}{5}C + 32$ (C)

$F - 32 = \frac{9}{5}C$

$5(F - 32) = 9C$

$? = C$

4. $A = \frac{h(B + b)}{2}$ (h,B)

$2A = h(B + b)$

$? = h$

5. $t = \frac{1}{2}(D - d)$ (D)

6. $E = IR$ (R)

7. $A = P + Prt$ (t)

8. $A = P(1 + rt)$ (P, r)

9. $A = 2\pi r^2 + 2\pi rh$ (h)

10. $PV = rT$ (V, r)

11. $C = \frac{5}{9}(F - 32)$ (F)

12. $F = ma$ (a)

13. $2S = n(a + b)$ (a,n)

$\frac{2S}{n} = a + l$

$? = a$

14. $C = 20d + 4n$ (d)

$20d = C - 4n$

$d = ?$

15. $L = 2\pi rh$ (r)

16. $K = \frac{WV^2}{2g}$ (V,W,g)

17. $S = at - \frac{1}{2}gt^2$ (a,g)

18. $T = \frac{E}{R + r}$ (E,R)

19. Calculate the unknown variable in each case.
(a) $S = 6e^2$, $e = 5$. (b) $C = 20 + 40(n - 1)$, $n = 8$.

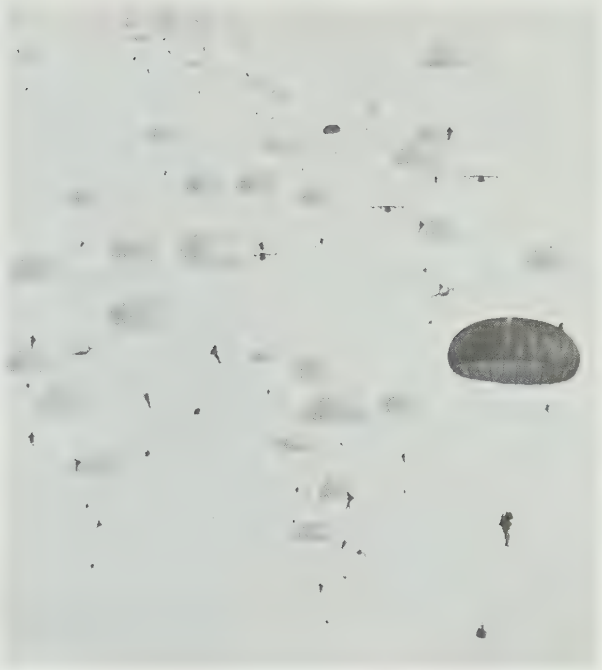
20. The formula used to find the area of a trapezoid is $A = \frac{h}{2}(a + b)$ when h is the measure of the altitude, and a and b are the measures of the lengths of the lower and upper bases of the trapezoid. If the area of the trapezoid is 108 square inches, the height 9 inches, and the lower base 12 inches, find the upper base.

21. The horsepower of a certain small motor is given by the formula $H = \frac{\pi dfs}{33{,}000}$.
(a) Find H when $s = 2800$, $f = 5$ and $d = 0.4$.
(b) Find d given $H = 0.6$, $f = 5.5$, $s = 2200$.

22. In the formula $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$:
(a) Solve for f given $p = 3$, $q = 4$.
(b) Solve for p, given $f = 5$, $q = 8$.
(c) Solve for p, given $f = 8$, $q = 6$.

23. (a) Use the formula in Example 1 to obtain the velocity of a satellite that completes an orbit every 24 hours at an altitude of 30,000 miles.
(b) Use the formula in Example 1 to obtain the time of one revolution of a satellite if its velocity is 8000 miles per hour at a height of 40,000 miles.

24. The distance S in feet that a body moves from rest while under constant acceleration of a feet per second per second for t seconds is given by
$$S = \tfrac{1}{2}at^2.$$
(a) Calculate the distance travelled in 15 seconds by a body starting from rest if the acceleration is 40 feet per second per second.
(b) How long does it take a body starting from rest to travel 300 feet if the constant acceleration is 30 feet per second per second?



(c) What must the constant acceleration be if a body travels 1500 feet in its first 15 seconds of movement?

25. If you drop a ball from the top of a high building the force of the earth's gravity causes the ball to accelerate at 32 feet per second per second. The top of the Husky Tower in Calgary is 626 feet above the street level. If a ball were dropped from the top of the structure how long would it take to strike the ground?

26. The tallest tower in the world is the Moscow Tower in the U.S.S.R. A weight dropped from the top of the structure takes approximately 10.4 seconds to strike the ground. Find the height of the tower.

27. The velocity v miles per second that a satellite must have to remain in orbit at a height h miles above the earth's surface is given by
$$v = \sqrt{\frac{GM}{3960 + h}}$$
when the product GM is 9.56×10^4. Find the velocity if the satellite is 300 miles above the earth's surface.

5·3 Order of Operations

The following chart reviews important terminology.

Terms	*Examples*	*Comments*
expression	$3x$, $5y$, $7z - 2y$, $4 - 3y^2 + 2x$.	A set of numbers and variables combined by any of the four operations: $+, -, \times, \div$.
numerical factor	'3' in $3x$	The numerical part of a product.
literal factor	'x' in $3x$	The *letter* part of a product.
terms	$3x$, $5y$, $7z$, 4, $-3y^2$	A number, a variable, or the product of a numerical and a literal factor.
polynomial	$3x - 4y + 5z - 3$, $x + y$, 3, y^2	The sums and/or differences of any number of terms.
monomial	$3x$, $\frac{4}{5}y$, 2, πr	One product or quotient (a polynomial of one term).
binomial	$3x - 4$, $\frac{2}{5}y + 7$, $6 - \frac{1}{3}z^2$	The sum or difference of two monomials (a polynomial of two terms).
trinomial	$3x - 4 + 5z$, $2y^2 + 4x^2 + 3$	The sums and/or differences of three monomials (a polynomial of three terms).
like terms	$\{3x, 4x, \sqrt{2}x\}$ $\{\pi x^3, 0.5x^3, 14x^3\}$ $\{2y^2, 5y^2, 6y^2\}$	The literal factors of each term in a set are the same and have the same exponent.
unlike terms	$3x^2$, $4x$, $\sqrt{2}y$, $3z$, $5w$, $\frac{6}{5}r$, $\frac{2t}{3}$	The literal factors of the terms are not the same and/or the exponents are not the same.

The following section reviews the order of operations. Attempt each example. If you need help read the rule for that question. The answer is given to each.

Example 1

The value of $7 \times 5 + 8 \div 2^2$ is

(a) $10\frac{3}{4}$ (b) 37 (c) $22\frac{3}{4}$ (d) 49.

Unless we have some rules to follow in this type of exercise in which grouping signs are omitted, each answer can be considered correct. The rules to use in this case are given.

Rule 1

When grouping signs are omitted
1. Simplify the powers.
2. Simplify the products and quotients.
3. Simplify the sums and differences.

Answer: (b) 37.

Example 2

The value of $7 \times (5 + 8) \div 2^2$ is

(a) $10\frac{3}{4}$ (b) 37 (c) $22\frac{3}{4}$ (d) 49

Rule 2

When grouping signs are used
1. Simplify the numbers (expression) *in* the grouping symbols first.
2. Then apply Rule 1.

Answer: (c) $22\frac{3}{4}$.

Example 3

The value of $[\,(7 \times 5) + 8\,] \div 2^2$ is

(a) $10\frac{3}{4}$ (b) 37 (c) $22\frac{3}{4}$ (d) 49.

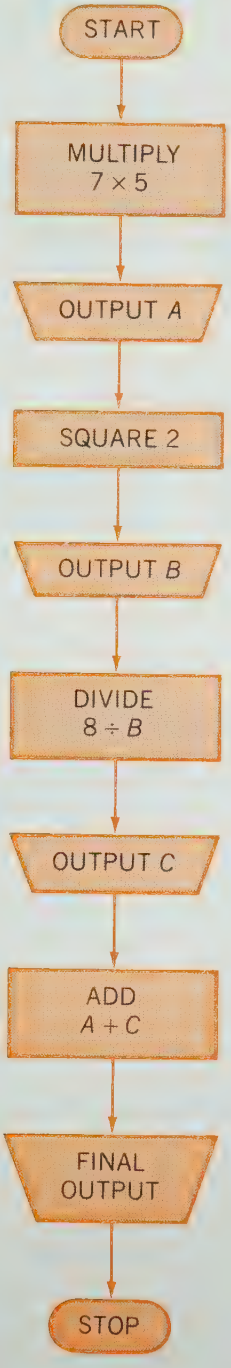
Rule 3

When grouping symbols are used inside other grouping signs.
1. Simplify the numbers (expression) in the innermost set of grouping signs first.
2. Simplify the number in the next outer set of grouping signs.
3. Continue simplifying until all grouping signs are removed.
4. Apply Rule 1.

Answer: (a) $10\frac{3}{4}$.

The order of operation in each example above can be illustrated by flow charts.

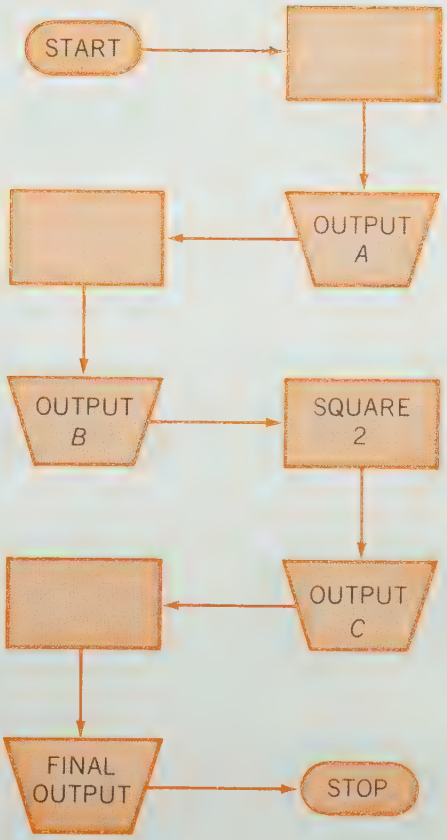
Example 1 is shown here:



EXERCISE 5·3

For each set, list the name(s) from the chart on page 78 that applies:

1. $\{5y, 2r, \sqrt{3}s\}$
2. $\{3y + 2y^2, 4y - 6z, 6 - 7p\}$
3. $\{x, m, n, k, l\}$
4. $\{4x^2 + 3y + 2, 5h + 6i - 2j, 6k - 2 + 2m\}$
5. $\{4r, 3r, \sqrt{5}r, \pi r\}$
6. $\{4x^2 + 4, 3y + 2y^2 + 4, 4r\}$
7. How would you group $7 \times 5 + 8 \div 2^2$ in order to get an answer of 49?
8. Complete this flow chart for Example 2. (This is just one of the acceptable solutions.)

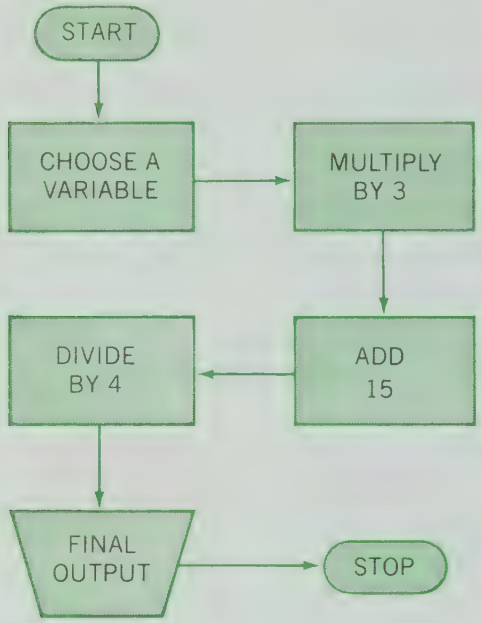


9. Draw a flow chart for Example 3.

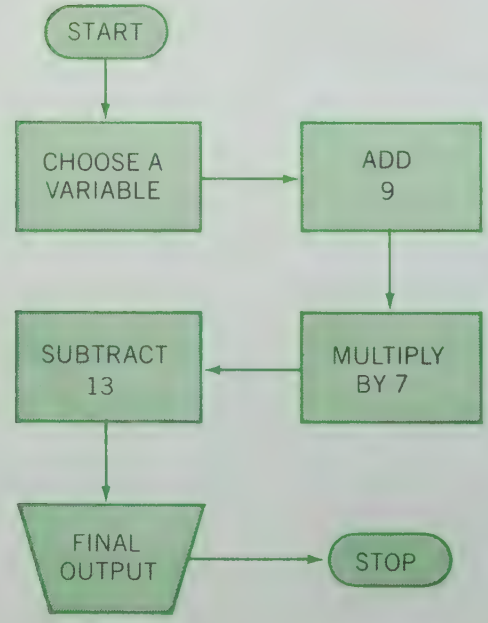
10. Draw a flow chart for your answer to question 7.

11. Follow the directions set out in each flow chart. Record the final output in each case.

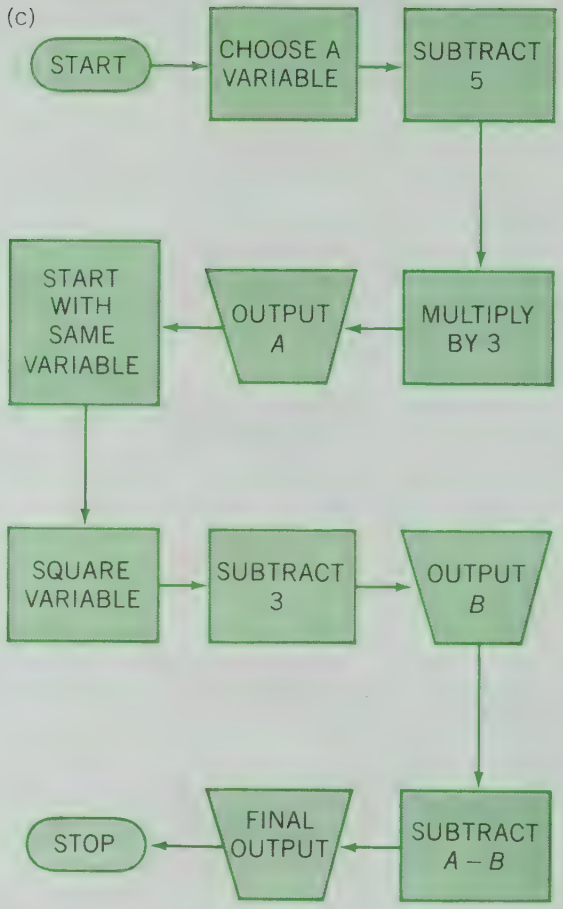
(a)



(b)



(c)



Evaluate each:

12. $3 + 4 \div 2 + 5$

13. $(3 + 4) \div (2 + 5)$

14. $7(9 \div 2) - 3$

15. $5(6 - 2) \div 4[3 \div (6 + 2)]$

16. $2[3 + (6 \div 4)] + 3[2 - (5 \div 4)]$

17. $3[y + 2(y - 2)] - 2[y - 3(y - 4)]$

18. $y(2 - 6) - 2[y - 3(y - 4)]$

19. $32[y - \frac{1}{2}(y - 2) - 2] - 3[y - \frac{1}{2}(3 - 2y)]$

20. $y(8 - 4) - 3[y - \frac{5}{6}(y + \pi)]$

21. $\frac{2}{3}(x - 4) - \frac{4}{9}[\frac{3}{2}x - \frac{1}{3}(2x + \frac{1}{9})]$

5·4 Addition and Subtraction of Polynomials

In each set of questions do the first three questions. If you have no difficulties go on to the next set. If you have difficulty do more of the questions in the set until you are certain you have no more difficulties.

Set I Simplify.

Addition

1. $(6x - 2y) + (-4x - 99y)$
2. $(3x^2 + 4y + 3x - 2y^2) + (13y - 2x^2 - 4y^2) + (3x - 4y^2 + 2x^2)$
3. $(5m + n - \sqrt{2}w) + (\pi m - 3n - 2w) + (4w - 6m + 2n)$
4. $(4m - 3n) + (-5m - 4n)$
5. $(4s) + (\pi s)$
6. $(4s + 5t) + (\sqrt{2}s - \sqrt{3}t)$
7. $(33b) + (-\sqrt{2}b) + (\pi b)$
8. $(55x - 32y - 5b^2) + (4b^2 - \pi y - \sqrt{2}x) + (\sqrt{3}b^2 - 4y)$

Set II Simplify.

Subtraction

1. $(4x - 2y) - (3y + 2x)$
2. $(47.5w - 22.3z) - (-4.1w - 51.8z)$
3. $(52.1m - \pi n) - (\sqrt{2}m - 4n) - (\sqrt{3}m - 2.75n + 4)$
4. $(4.7d) - (2.5d)$
5. $(22b) - (-15b)$
6. $(56.4d) - (\pi d)$
7. $(251.4m - 591.7n + 32.9p) - (200.4m - 501.9n + 33.1p)$
8. $(51xy) - (22xy - 17yz) - (23zy + 29xz - 29xy) - (2zy + 3xz - 4xy)$
9. $(22xy - 5zr) - (23xw - 27zw) - 3zr$

5·5 Using the Distributive Property in Algebra

The distributive property occurs over and over again in algebra. Its use is illustrated in this section. The measure of the area of the rectangle *XYZW* in Figure 5-1 is

$$a(b + c) = ab + ac.$$

The measure of the area of the rectangle *MNOP* in Figure 5-1 is

$$h(g + d + f) = hg + hd + hf.$$

We can use these illustrations as a model of the distributive property.

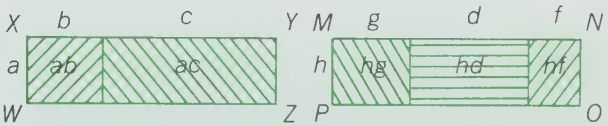


Figure 5-1

Example 1

Perform the indicated multiplication.

(a) $ax(bx + cy)$ (b) $s(t + v + m + r)$

Solution

(a) $ax(bx + cy) = axbx + axcy$
$= abx^2 + axcy$

(b) $s(t + v + m + r) = st + sv + sm + sr$

The distributive property is used in reverse to the illustrations in Example 1 to factor certain binomials. Continuing with our model, study the rectangle in Figure 5-2. We see the measure of the area of the rectangle $ABCD$ is $(sm + sn)$. What is the length? What is the width?

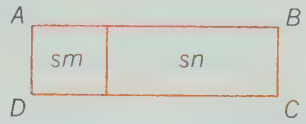
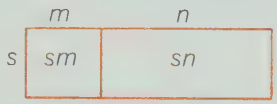


Figure 5-2

$$\text{Area} = \text{width} \times \text{length}$$
$$sm + sn = (?)(?)$$
$$sm + sn = s(m + n)$$

In this illustration $(sm + sn)$ has been factored into s and $(m + n)$. We say "s is the *common factor* of the expression $sm + sn$."



Example 2

Factor

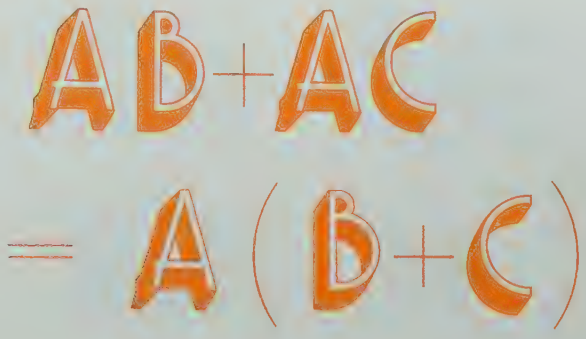
(a) $mnb + msn$ (b) $wv + 2wz + 3wy$ (c) $36y^2 + 30x$

Solution

(a) $mnb + msn = mn(b + s)$ common factor is mn.

(b) $wv + 2wz + 3wy = w(v + 2z + 3y)$ common factor is w.

(c) $36y^2 + 30x = 6 \cdot 6y^2 + 6 \cdot 5x$ common factor is 6.
$= 6(6y^2 + 5x)$



The product of two binomials is an extension of the distributive property. Observe this illustration.

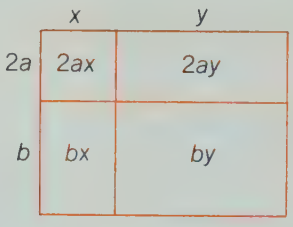


Figure 5-3

In Figure 5-3.

$$(2a + b)(x + y) = \ ?$$

From our model we know the answer is
$2ax + 2ay + bx + by$.
Let us use the distributive property to expand $(2a + b)(x + y)$.

$(2a + b)(x + y)$

$$(2a + b)\boxed{(x + y)} = 2a\boxed{(x + y)} + b\boxed{(x + y)}$$

Treat $(x + y)$ as one number.

$$= 2ax + 2ay + bx + by$$

Example 3

Perform the indicated multiplication.

(a) $(2a + 3b)(6a - 2b)$

(b) $(x - 2y)(x - 2y)$

Solution

(a) $(2a + 3b)\boxed{(6a - 2b)} = 2a\boxed{(6a - 2b)}$
$+ 3b\boxed{(6a - 2b)}$
$= 12a^2 - 4ab + 18ba - 6b^2$
$= 12a^2 + 14ab - 6b^2$

(b) $(x - 2y)\boxed{(x - 2y)} = x\boxed{(x - 2y)} - 2y\boxed{(x - 2y)}$
$= x^2 - 2xy - 2yx + 4y^2$
$= x^2 - 4xy + 4y^2$

We shall now look at a short cut to multiplying two binomials. Do you remember the FOIL Rule?

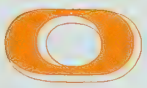
Example 4

Expand:

(a) $(2x + y)(4x + 3y)$
(b) $(3m - 2n)(5m + 8n)$

Solution

(a) $(2x + y)(4x + 3y) = 2x \cdot 4x + \ldots$



$(2x + y)(4x + 3y) = 2x \cdot 4x + 2x \cdot 3y + \ldots$

$(2x + y)(4x + 3y) = 8x^2 + 6xy + y \cdot 4x + \ldots$

$(2x + y)(4x + 3y) = 8x^2 + 6xy + 4yx + 3y^2$
$= 8x^2 + 10xy + 3y^2$

(b) $(3m - 2n)(5m + 8n)$
$= 3m \cdot 5m + 3m \cdot 8n + (-2n)(5m) + (-2n)(8n)$
$= 15m^2 + 24mn - 10nm - 16n^2$
$= 15m^2 + 14mn - 16n^2$

EXERCISE 5·5

1. Using a model similar to those used in this section, draw a figure to show the product of:

(a) $a(a + b)$	(b) $x(r + t)$
(c) $2a(2a + b)$	(d) $(a + b)(a + b)$
(e) $(m + n)(2m + n)$	(f) $(x + 2y)(2x + y)$
(g) $(2a + b)^2$	(h) $(s + 3t)^2$

2. Find the width of each rectangle from the data shown.

(a)

?	3v	6
	6v	12

(b)

?	4w	3s
	12wr	9sr

(c)

?	a	a	d
	ab	ab	bd
	ab	ab	bd
	ca	ca	cd

(d)

?	x	y	y
	xm	ym	ym
	xm	ym	ym
	x^2	xy	xy

3. Perform the indicated multiplication.

(a) $h(a + t)$	(b) $4y(3t + 2s)$
(c) $16(2y - 3x)$	(d) $\frac{3}{4}y(\frac{8}{9}y - 12z)$
(e) $-3 \cdot 6m^2(m^2 - 6n^3)$	(f) $-s(s^2 - s^3)$
(g) $-b^3(-b^2 - b - 3c + 2d)$	(h) $4z(\frac{3}{8}z^2 + \frac{1}{16}z - \frac{1}{4})$
(i) $\frac{5w}{2}(w^2 - 8w - 2)$	(j) $\frac{4r}{7s}(\frac{5s}{3r} - \frac{49r}{3s^2} + \frac{16s^2}{5r^2})$

4. Multiply:

(a) $(a - 2b)(a + 2b)$	(b) $(3x + y)(m + 2n)$
(c) $(d + r)(2d + 3r)$	(d) $(5w - 2y)(5w - 2y)$
(e) $(14a - 3d)(14a - 3d)$	(f) $(w^2 - v)(w^2 - v)$
(g) $(a - 2b)(a + 2b)$	(h) $(3a - 5b)(3a + 5b)$
(i) $(2d - 4g)(2d + 4g)$	(j) $(y - y^2)(y - y^2)$
(k) $(a + b)^2$	(l) $(m + 2n)^2$

5. Factor:

(a) $ab + ac + ad$	(b) $2xy + 2xz - 4xw$
(c) $\frac{3}{2}x + \frac{3}{2}b$	(d) $\frac{4}{5}y - \frac{8}{5}y^2$
(e) $\frac{4}{3}a^2 - \frac{2}{3}b^2a$	(f) $\frac{5}{9}y^3 - \frac{5}{9}y^2 - \frac{5}{9}y$
(g) $3x^2y^2 + 3x^3y - 3x^2y$	(h) $8k^4 + 16k^3t + 24k^2$

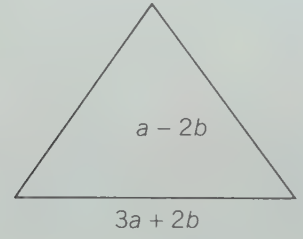
6. If the width of a rectangle is doubled but the length remains the same, how does the area change?

7. If the length and width of a rectangle are doubled, how does the area change?

8. A student works in the cafeteria d days at m dollars a day. How much money does he earn? A second student works 4 days more than the first student at three dollars per day more than the first. How much does the second student earn?

9. An Oldsmobile travels x hours at y miles per hour. A Buick travels twice as long at 20 miles per hour faster than the Oldsmobile. How much farther did the Buick travel than the Oldsmobile?

10. A triangle has the dimensions shown below. What is the measure of its area?



11. A land developer holds an option on $(4x^2 + 6yx)$ square rods of land in the shape of a rectangle. What are some of the things you can tell about the possible dimensions of the parcel of land?

5·6 Special Products: Squares of Sums and Differences

Short methods of finding the products of the square of a sum and the square of a difference will be illustrated. We will be working with binomials in two variables.

Example 1

Find the product of each.

(a) $(3x + 4y)^2$ (b) $(ax + by)^2$ (c) $(6x + 4y)^2$

Solution

Do you remember the FOIL Rule?

(a) $(3x + 4y)^2 = (3x + 4y)(3x + 4y)$
$= 9x^2 + 12xy + 12yx + 16y^2$
F O I L
$= 9x^2 + 24xy + 16y^2$

(b) $(ax + by)^2 = (ax + by)(ax + by)$
$= a^2x^2 + axby + byax + b^2y^2$
F O I L
$= a^2x^2 + 2(axby) + b^2y^2$
or $= a^2x^2 + b^2y^2 + 2(axby)$

We can write this as a rule:

> The square of the sum of two terms is the sum of the squares of each of the two terms and twice the product of the two terms.

(c) Now let us use this rule to do this problem:

$(6x + 4y)^2 = 36x^2 + 16y^2 + 2(24xy)$
$= 36x^2 + 16y^2 + 48xy$

Example 2

Find the product of each:

(a) $(3m - 4n)^2$ (b) $(ax - by)^2$

Solution

(a) $(3m - 4n)^2 = [3m + (-4)]^2$ Express as the sum of two terms.

$= (3m)^2 + (-4n)^2 + 2(3m)(-4n)$ Apply the rule.

$= 9m^2 + 16n^2 - 24mn$

(b) $(ax - by)^2 = [ax + (-by)]^2$ Express as the sum of two terms.

$= (ax)^2 + (-by)^2 + 2(ax)(-by)$ Apply the rule.

$= a^2x^2 + b^2y^2 - 2axby$

EXERCISE 5·6

Write the product of each of the following using the rules established above. Try to do each in one step.

1. $(x + y)^2$
2. $(x + 2y)^2$
3. $(2x + 3y)^2$
4. $(3x + 2y)^2$
5. $(8x + 4y)^2$
6. $(10x + 12y)^2$
7. $(\frac{2}{3}x + \frac{1}{4}y)^2$
8. $(\frac{3}{4}x + \frac{3}{5}y)^2$
9. $(x - y)^2$
10. $(x - 2y)^2$
11. $(3x - 4y)^2$
12. $(4x - 9y)^2$
13. $(5x - \frac{1}{4}y)^2$
14. $(6x - \frac{2}{3}y)^2$
15. $(\frac{2}{3}y - \frac{1}{5}z)^2$

Find the square of each binomial.

16. $\frac{3x}{4} - \frac{y}{2}$
17. $\frac{x}{3} + \frac{y}{4}$
18. $\frac{3ab}{c} + \frac{4xy}{z}$
19. $\frac{2mn}{3} - \frac{3gh}{4}$
20. The side of a square parking lot is $(4x + 5y)$ yards. Find the area of the lot.
21. A square has a side $(\frac{2}{3}x - \frac{4}{5}y)$ feet. Find the area of the square.
22. The force of a moving body is given by $F = mv^2$. Find F if $m = 32$ and $v = 8x + 3y$.
23. Find the area of a circle given $r = 12x - 3y$.
24. The number 99 may be written as $100 - 1$. Using this fact, evaluate 99^2.
25. The number 101 may be written as $100 + 1$. Using this fact, evaluate 101^2.
26. Use the method indicated in questions 24 and 25 to evaluate:
    (a) 49^2 (b) 51^2 (c) 999^2 (d) 1001^2
    (e) 501^2 (f) 499^2 (g) 399^2 (h) 199^2

5·7 Special Product: (a + b)(a − b)

We will apply the procedure used in the previous section.

Example 1

Find the product of each.

(a) $(2m + 3n)(2m - 3n)$ (b) $(ax + by)(ax - by)$

(c) $(4w + 5v)(4w - 5v)$

Solution

(a) $(2m + 3n)(2m - 3n) = 4m^2 - 2m\,3n + 3n\,2m - 9n^2$
F O I L

$= 4m^2 - 6mn + 6mn - 9n^2$

$= 4m^2 - 9n^2$

(b) $(ax + by)(ax - by) = a^2x^2 - axby + byax - b^2y^2$

$= a^2x^2 - b^2y^2$

The Rule

> The product of one binomial (the sum of two terms) and another binomial (the difference of the same two terms) is the square of the first term less the square of the second term.

(c) Let us use this rule:

$$(4w + 5v)(4w - 5v) = 16w^2 - 25v^2$$

square of *less* square of
1st term 2nd term

EXERCISE 5·7

Use the rule to find the product in each. Endeavour to do each in one step.

1. $(x+y)(x-y)$
2. $(2x-y)(2x+y)$
3. $(3x-y)(3x+y)$
4. $(5x-4y)(5x+4y)$
5. $(8x-9y)(8x+9y)$
6. $(ax+by)(ax-by)$
7. $\left(\frac{2x}{3}+\frac{5y}{4}\right)\left(\frac{2x}{3}-\frac{5y}{4}\right)$
8. $\left(\frac{\pi x}{2}-\frac{3y}{2}\right)\left(\frac{\pi x}{2}+\frac{3y}{2}\right)$
9. $\left(\frac{7x}{9}+\frac{2y}{5}\right)\left(\frac{7x}{9}-\frac{2y}{5}\right)$
10. $\left(\frac{x}{3}-\frac{y}{5}\right)\left(\frac{x}{3}+\frac{y}{5}\right)$
11. $(abc-mn)(abc+mn)$
12. $(3rst+4klm)(3rst-4klm)$
13. $(2ut-4rt)(2ut+4rt)$
14. $(10vw-13ut^2)(10vw+13ut^2)$
15. $(x^2y^3-3z^2)(x^2y^3+3z^2)$
16. $(21kt-6rg)(21kt+6rg)$

5·8 Factoring

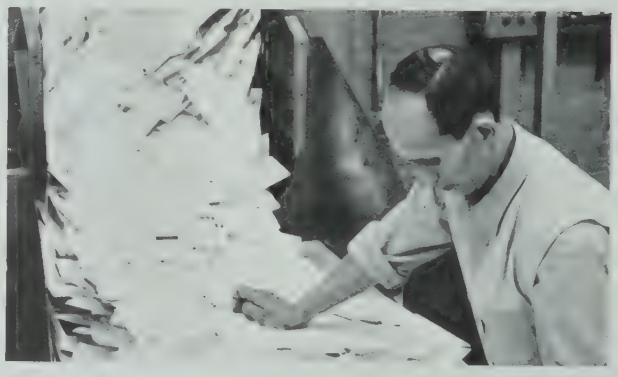
In a previous section we expressed the product of two terms as one term, and the product of two polynomials as one polynomial. For example, $3x \cdot 2y$ may be written $6xy$ and $(x-1)(x+1)$ is x^2-1. It is often desirable to express a term as the product of two terms or a polynomial as the product of two polynomials. Each term can be written as the product of at least two terms. For example, 19 can be written as 19×1 and x can be written as $x \times 1$. This operation of expressing a term (or a polynomial) as a product of two or more terms (or polynomials) is called *factoring*. Each term in the expanded product is a *factor* of the original term.

$$6x^2 = (2x)(3x) \qquad 5x = (5)(x)$$
$$= (6)(x^2) \qquad = (1)(5x)$$
$$= (1)(6x^2)$$

You will notice from the two instances that terms may have more than one set of factors.

Some polynomials can be factored. That is, some polynomials can be expressed as the product of two or more polynomials. Study the set of polynomials and the factored forms.

Polynomial	Factored form
1. a^2+ab	$a(a+b)$
2. $3x+6y$	$3(x+2y)$
3. $4a^2-b^2$	$(2a-b)(2a+b)$
4. m^2-9n^2	$(m-3n)(m+3n)$
5. $a^2+6ab+9b^2$	$(a+3b)(a+3b)$
6. $3t^2+8tv-3v^2$	$(3t-v)(t+3v)$



In what sense is this man factoring?

EXERCISE 5·8

1. Refer to the column "Factored Form". Multiply each. Do you get the corresponding polynomial shown in the left column?

2. Fill in the multiplication table with the appropriate products. (Draw the table full page size in your exercise book.)

	$a+3b$	$a+2b$	$a+b$	a	$a-b$	$a-2b$	$a-3b$
$a-3b$							
$a-2b$		a^2-4b^2					
$a-b$					0		
a				a^2			a^2-3ab
$a+b$			4				
$a+2b$							
$a+3b$				4			

(a) Notice that the sums of the coefficients of some of the expressions have been placed in color in the appropriate squares.

$$a^2 - 2ab + b^2$$

has a coefficient sum of

$$1 - 2 + 1 = 0.$$

Place in color the sums of the coefficients of the expression.

(b) What patterns do you see with reference to the sums of the coefficients of the factors and of the products?

(c) What use can you see for these patterns? Make a rule about coefficients.

(d) What do you notice about the products on the diagonal from the lower left to the upper right?

(e) What pattern do you notice about the products at the right end of each row?

(f) If you were looking for the factors of $a^2 + 8ab + 15b^2$, which of the following would you consider? Why?

(i) $(a + 11b)(a + 4b)$
(ii) $(a - 5b)(a + 3b)$
(iii) $(a + 15b)(a + 1b)$
(iv) $(a + 5b)(a + 3b)$
(v) $(a + 4b + b)(a + 4b - b)$
(vi) $(a - 5b)(a - 4b)$

(g) Using the *coefficient rule* established in (c), check each of these multiplications. Which are correct?

1. $(a + 3b)(a - 2b) = a^2 + 5ab - 6b^2$
2. $(a - 4b)(a - 2b) = a^2 - 6ab + 8b^2$
3. $(a - 5b)(a + 2b) = a^2 - 7ab - 10b^2$
4. $(a + 7b)(a + 3b) = a^2 + 10ab - 21b^2$
5. $(a - 4b)(a + 4b) = a^2 - b^2$

(h) Do you think this *coefficient rule* applies in cases like these?

(i) $(2a - 3b)(3a + 2b)$
(ii) $(3a + 5b)(4a - 6b)$
(iii) $(9a - 7b)(8a - 6b)$
(iv) $(6a + 5b)(6a - 5b)$

Find the product in each and apply the *coefficient rule* to see if it is true for these cases.

5·9 A Common Factor

We shall now look at various types of factoring in turn. In the first type all terms of the polynomials have a common factor. We use the distributive property in reverse.

Example 1

Factor each.

(a) $30m + 12n$
(b) $7m^3 - 9m^2$
(c) $9a^2b^2c - 3ab^2c^3 + 27ab^2c^2$
(d) $3x(y + 1) - 2z(y + 1)$

Solution

(a) $30m + 12n = 6 \cdot 5m + 6 \cdot 2n$

The highest factor common to each term is 6.

$$30m + 12n = 6(5m) + 6(2n)$$
$$= 6(5m + 2n)$$

(b) $7m^3 - 9m^2$

The highest factor common to each term is m^2.

$$7m^3 - 9m^2 = 7m \cdot m^2 - 9 \cdot m^2$$
$$= m^2(7m - 9)$$

(c) The factor common to each term here is a little harder to find. First, consider the numerical factor of each term. The common factor of all the numerical factors is 3. Then, consider each of the literal variables in turn. Each has a factor a; each has a factor b^2; each has a factor c. Therefore, we can write

$$9a^2b^2c - 3ab^2c^3 + 27ab^2c^2$$
$$= 3 \cdot a \cdot b^2 \cdot c \cdot 3a - 3 \cdot a \cdot b^2 \cdot c \cdot c^2 + 3a \cdot b^2 \cdot c \cdot 9 \cdot c$$
$$= 3ab^2c(3a - c^2 + 9c)$$

(d) The common factor in this one is $(y + 1)$.

$$3x(y + 1) - 2z(y + 1) = (y + 1)(3x - 2z)$$

The student should always check his factoring by expanding his answer to see that he gets the polynomial given in the question. Note how we check part (a) of Example 1.

$$6(5m + 2n) = 30m + 12n \quad \checkmark \text{ Checks.}$$

Rational expressions can often be simplified. In the examples that follow we will factor the numerator, denominator or both, then simplify.

Example 2

Simplify

(a) $\dfrac{a^2b - ab^2}{ab}$ (b) $\dfrac{2x^2y - 4xy^2}{8xy^2 + 18x^2y}$

Solution

(a) $$\begin{aligned}\frac{a^2b - ab^2}{ab} &= \frac{ab(a-b)}{ab}\\ &= \frac{ab}{ab}(a-b)\\ &= 1(a-b)\\ &= a-b.\end{aligned}$$

(b) $$\begin{aligned}\frac{2x^2y - 4xy^2}{8xy^2 + 18x^2y} &= \frac{2xy(x-2y)}{2xy(4y+9x)}\\ &= 1\left(\frac{x-2y}{4y+9x}\right)\\ &= \frac{x-2y}{4y+9x}\end{aligned}$$

EXERCISE 5·9

1. Check each of the questions in Example 1, parts (b), (c), and (d).

Factor and check each of the following. Do the first 5. If you have difficulty do the next 5. If you do not have difficulty with the first 5 do the last 5.

2. $33m^2 - 11m$
3. $42a^2b^2 - 7ab^2$
4. $12rs - 7rt$
5. $ab^2 - 3b^3$
6. $24xy^3z - 6xyz^2 + 48y^2z$
7. $15xw^2 - 9w$
8. $5m(n-w) - 3p(n-w)$
9. $24xz^2 - 15x^2 + 18xz$
10. $3x(y-2z) + 5(y-2z)$
11. $xy(z-w) + 7l(z-w)$
12. $14x^2y^3 - 28x^2y + 49x^3y^2$
13. $xw(z^2-w) + x^2w^3(z^2-w)$
14. $5xy - 10x^2y^2 + 20x^3y^3$
15. $z^2(3a-2b+c) - zw^2(3a-2b+c) + zw(3a-2b+c)$
16. $xy + zy + xw + zw + y^2 + yw$
17. Explore to find a relation between the two prime factors of a positive integer which has only two prime factors, and the sum of *all* the factors of that integer.
    Hint! Choose any positive integer which has only two prime factors, for example, 15. Write *all* the factors of the integer: (3, 5, 1, 15). Find the sum of the factors. Now look at the two prime factors of the number (3, 5). How can you relate them to the sum? Use several different integers to test your theory.

Simplify the following.

18. $\dfrac{xy - xz}{x}$
19. $\dfrac{yb - y^2c}{yb}$
20. $\dfrac{ab - abc}{ab}$
21. $\dfrac{x^2y^3 - x^2y^2}{x^2y^2}$
22. $\dfrac{xy + xz + xw}{xy}$
23. $\dfrac{3x^2 - 6x^2y}{3x^2y^2}$
24. $\dfrac{8a^2b - 16ab^2}{2ab^2 - 2abc}$
25. $\dfrac{x^2y - xy^2}{y^2x - x^2y} \times \dfrac{y}{x}$
26. $\dfrac{ram - ran}{rbm + rbn} \times \dfrac{cbm + cbn}{cam - can}$
27. $\dfrac{mars + rsth}{rsf + rsun} \times \dfrac{mi + ms}{me + mh}$

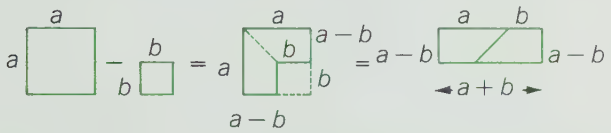
5·10 Factoring: Difference of Squares

$a^2 - b^2 = (a - b)(a + b)$

We saw earlier that $(a - b)(a + b) = a^2 - b^2$.
We can reverse the statement and write

$$a^2 - b^2 = (a - b)(a + b)$$

We say we have factored the expression $a^2 - b^2$ into $(a - b)$ and $(a + b)$. We can indicate this graphically.



$a^2 - b^2 = (a - b)(a + b)$

The difference of the squares of two monomials is the product of the difference of the two monomials and the sum of the two monomials.

We can use this type of factoring to good advantage in science and mathematics when the computation involves the difference of two squares.

Example 1

Factor each:

(a) $9a^2 - 4b^2$ (b) $16a^2 - 25y^2$.

Solution

(a) $9a^2 - 4b^2 = (\ ?\ )^2 - (\ ?\ )^2$

We want to write this as the difference of the squares of two monomials. Therefore, the square root of $9a^2$ ($\sqrt{9a^2}$) goes in the first set of parentheses while the square root of $4b^2$ ($\sqrt{4b^2}$) goes in the second set.

$\because \sqrt{9a^2} = 3a, \quad \sqrt{4b^2} = 2b$

$\therefore \quad 9a^2 - 4b^2 = (3a)^2 - (2b)^2$

$9a^2 - 4b^2 = (3a - 2b)(3a + 2b)$

$9a^2 - 4b^2$: difference of the squares of two monomials

$(3a - 2b)$: difference of the two monomials

$(3a + 2b)$: sum of the two monomials

$(3a - 2b)(3a + 2b)$: product

(b) $16x^2 - 25y^2 = (\ ?\ )^2 - (\ ?\ )^2$

$\because \quad \sqrt{16x^2} = 4x$ and $\sqrt{25y^2} = 5y$

$\therefore \quad 16x^2 - 25y^2 = (4x)^2 - (5y)^2$

$= (4x - 5y)(4x + 5y)$

Example 2

Evaluate $19^2 - 16^2$.

Solution

The usual way to do this problem would be to square 19 and square 16 then subtract the latter from the first. This involves considerable computation unless tables or a slide rule is available. Notice how easily it is done by using the *difference of squares* method.

$19^2 - 16^2 = (19 - 16)(19 + 16)$ (Factor.)

$= 3(35)$

$= 105.$

EXERCISE 5·10

Express each in factored form. Endeavour to do as many as you can in one step by using the rule.

1. $x^2 - y^2$
2. $m^2 - n^2$
3. $4b^2 - 25d^2$
4. $9k^2 - 16h^2$
5. $36z^2 - 49y^2$
6. $100y^2 - 144w^2$
7. $121r^2 - 9v^2$
8. $225t^2 - 36s^2$
9. $r^2t^2 - v^2u^2$
10. $64x^2 - 144y^2$

Use the "difference of squares" method to evaluate each.

11. $23^2 - 17^2$
12. $38^2 - 28^2$
13. $19^2 - 29^2$
14. $56^2 - 54^2$
15. $189^2 - 169^2$
16. $343^2 - 157^2$
17. $46^2 - 42^2$
18. $38^2 - 42^2$

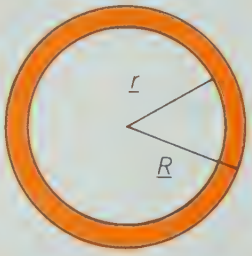
19. Check your answers to questions 11 through 14 by using tables or slide rule.

20. The derivation of the formula for the area of the shaded portion is shown:

Area = area of large circle
– area of small circle.

$A = \pi R^2 - \pi r^2$

$A = \pi(R^2 - r^2)$



Find the area given

(a) $R = 17$, $r = 14$

(b) $R = 18$, $r = 17$

(c) $R = 28$, $r = 26$.

21. A formula for a free falling body is $v^2 - u^2 = 2as$ where v is the final velocity, u is the initial velocity, a is the acceleration due to gravity and s is the distance fallen. In each of the following evaluate $v^2 - u^2$ then solve for the required unknown variable.

(a) Given: $v = 125$ ft/sec,
$u = 25$ ft/sec,
$a = 32$ ft/sec/sec

Find: s, the distance in feet the body fell.

(b) Given: A body with an initial velocity (u) of 20 ft/sec has a final velocity (v) of 180 ft/sec after falling 500 feet.

Find: a, the acceleration in feet per second per second.

(c) Given: $a = 32$, $v = 280$, $u = 20$. (All measures are in appropriate units.)

Find: s.

22. The difference in the amount of heat produced by an electrical heat element as the current increases is given by the formula

$$H = 0.24RC^2 - 0.24Rc^2$$
$$= 0.24R(C^2 - c^2)$$

where H is in calories, R is in ohms, C and c are in amperes.

(a) Express $H = 0.24RC^2 - 0.24Rc^2$ in complete factored form.

(b) Find the heat difference given $R = 50$ ohms, $C = 27$ amps, and $c = 18$ amps.

(c) Find H given $R = 48$ ohms, $C = 54$ amps, and $c = 16$ amps.

23. Factor each completely. Notice that each has two stages of factoring.

(a) $x^4 - 1$
(b) $x^4 - y^4$
(c) $81x^4 - y^4$
(d) $10000m^4 - 1$

24. Factor each:

(a) $(a - b)^2 - y^2$
(b) $(a + b)^2 - y^2$
(c) $(2r - s)^2 - t^2$
(d) $(m - n)^2 - h^2$
(e) $(3g + 4h)^2 - y^2$
(f) $(x + y)^2 - (z - w)^2$.

25. Simplify:

(a) $\dfrac{a^2 - b^2}{a - b}$
(b) $\dfrac{a^2 - 4b^2}{a + 2b}$
(c) $\dfrac{2x^2 - 8y^2}{2(x - 2y)}$
(d) $\dfrac{3x^2 - 27y^2}{3x - 9y}$.

26. Factor each:

(a) $18t^2 - 50r^2$
(b) $32m^2 - 8r^2$
(c) $27m^2 - 108n^2$
(d) $4mn^2 - 9mn^2$
(e) $20r^2 - 125s^2$
(f) $m^3n^5 - m^5n^3$.

5·11 Grouping to Factor

Often all the terms of a polynomial do not have a common factor. In some of these cases we can group the terms into two or more sets of several terms, each set having a common factor. Each set is dealt with separately.

Example 1

Group and factor this polynomial.

$$xa + xb + za + zb$$

Solution

We will group the terms that have x's and the terms that contain z's. Then we will factor out the common factor in the first group and in the second group separately.

$$\begin{aligned} xa + xb + za + zb &= (xa + xb) + (za + zb) \\ &= x(a + b) + z(a + b) \\ &= (a + b)(x + z) \end{aligned}$$

The order in which the terms occur does not always facilitate factoring by the common factor method. Therefore, it is often necessary to rearrange and then group in order to factor.

Example 2

Rearrange, group and factor:

$$4x - 3y^2 + 4y - 3xy.$$

Solution

$$\begin{aligned} 4x - 3y^2 + 4y - 3xy &= 4x + 4y - 3xy - 3y^2 \\ &= 4(x + y) - 3y(x + y) \\ &= (4 - 3y)(x + y) \end{aligned}$$

Notice we can rearrange and group differently.

$$\begin{aligned} 4x + 4y - 3xy - 3y^2 &= 4x - 3xy + 4y - 3y^2 \\ &= x(4 - 3y) + y(4 - 3y) \\ &= (x + y)(4 - 3y) \end{aligned}$$

The same answer!

EXERCISE 5·11

Discuss how you would rearrange and group to facilitate factoring in questions 1 through 5.

1. $4a - cb - 4b + ca$
2. $3m + mn + 3n + m^2$
3. $5a + ab + 5b + a^2$
4. $m + p^2 + p + pm$
5. $r + rs + s + r^2$
6. In how many other ways can Example 2 be grouped and factored? Confer with your classmates after you have done the example in as many ways as you can. Do your classmates have ways you don't? Do you always end up with the same answer?
7. Factor questions 1 through 5.

Factor each of the following.

8. $3a + 3b + ma + mb$
9. $3xa - 3xb + 2mn - 2mb$
10. $4m + 4n + 5am + 5an$
11. $4m + 5an + 4n + 5am$
12. $4r - 4s + 5tr - 5ts$
13. $mais + mafun + this + thfun$
14. $4rt + 80rp + 3pt + 60p^2$
15. $4rt + 60p^2 + 80rp + 3pt$
16. $24am + 16a^2m + 27b + 18ab$
17. $18ab + 24am + 27b + 16a^2m$
18. How many ways can you group question 11? question 15?
19. Simplify

    (a) $\dfrac{xy - xw + ay - aw}{x + a}$ (b) $\dfrac{mn - mr + sn - sr}{n - r}$

5·12 Division Using Polynomials

Earlier you practised multiplying a binomial by a monomial and by a binomial. Let us look carefully at the following multiplication in anticipation of the inverse operation to multiplication: division.

Recall that since $7 \times 8 = 56$, then $7 = \frac{56}{8}$ and $8 = \frac{56}{7}$.

Now since

$$(3a + 4b)(5a + 2b) = 15a^2 + 26ab + 8b^2$$

we know $$3a + 4b = \frac{15a^2 + 26ab + 8b^2}{5a + 2b}$$

and $$5a + 2b = \frac{12a^2 + 26ab + 8b^2}{3a + 4b}.$$

This merely indicates that we can divide a polynomial by a polynomial and get a quotient which is a polynomial. We will do this later. In the examples that follow, you will see that long division can be performed with polynomials in much the same manner as with the natural numbers. Recall that 672 is an abbreviation of $600 + 70 + 2$.

Study the parallel processes:

Problem $\frac{672}{32}$

Step 1	$32\overline{)\,672}$ $\underline{64}$ 3	$30+2\overline{)\,600+70+2}$ $\underline{600+40}$ $0+30$
Step 2	$32\overline{)\,672}$ $\underline{64}$ 32 $\underline{32}$ 0	$30+2\overline{)\,600+70+2}$ $\underline{600+40}$ $30+2$ $\underline{30+2}$ 0
Check	32 21 $\overline{32}$ 64 $\overline{672}$	$30+2$ $20+1$ $\overline{30+2}$ $600+40$ $\overline{600+70+2}$

Problem $\frac{5x + 30}{5}$

Step 1

$$5\overline{)\,5x+30} \quad (x)$$
$$\underline{5x}$$
$$0$$

Step 2

$$5\overline{)\,5x+30} \quad (x+6)$$
$$\underline{5x}$$
$$0+30$$
$$\underline{30}$$
$$0$$

Check

$$x + 6$$
$$\underline{5}$$
$$5x + 30$$

Problem $\frac{x^2 + 5x + 6}{x + 3}$

Step 1

$$x+3\overline{)\,x^2+5x+6} \quad (x)$$
$$\underline{x^2+3x}$$
$$0+2x$$

Step 2

$$x+3\overline{)\,x^2+5x+6} \quad (x+2)$$
$$\underline{x^2+3x}$$
$$2x+6$$
$$\underline{2x+6}$$
$$0$$

Check

$$x + 3$$
$$\underline{x + 2}$$
$$2x + 6$$
$$x^2 + 3x$$
$$\overline{x^2 + 5x + 6}$$

Now let us look in more detail at what we did in the long division.

Steps to Long Division

Step 1. $x + 3\overline{)x^2 + 5x + 6}$

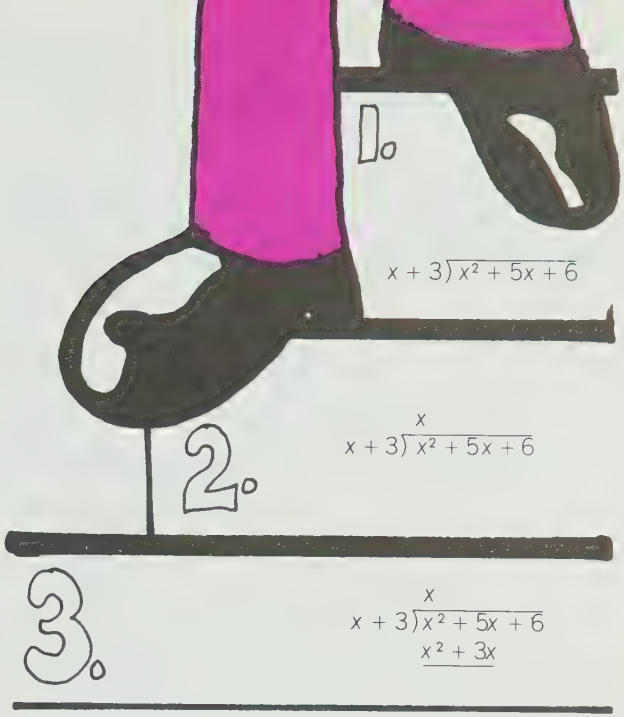
Notice we wrote the divisor and dividend in the normal long division arrangement.

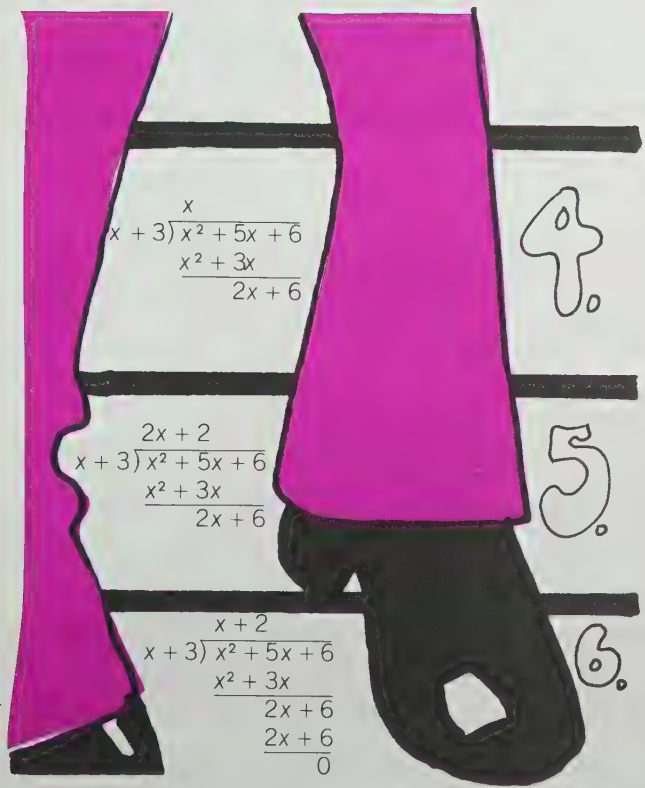
Step 2.
$$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \end{array}$$

Ask yourself "What must I multiply x by to get *exactly* x^2? The answer is x. This partial quotient is placed above the x^2.

Step 3.
$$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \end{array}$$

Now multiply the divisor $(x + 3)$ by the partial quotient x and place this product under the corresponding *like terms* of the dividend.


1.
$x + 3\overline{)x^2 + 5x + 6}$
2.
$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \end{array}$
3.
$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \end{array}$


4.
$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \end{array}$
5.
$\begin{array}{r} 2x + 2 \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \end{array}$
6.
$\begin{array}{r} x + 2 \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \\ \underline{2x + 6} \\ 0 \end{array}$

Step 4.
$$\begin{array}{r} x \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \end{array}$$

Subtract this product from the dividend. The remainder is the new dividend. Is the degree of the remainder less than the degree of the divisor? If *No*, continue. If *Yes*, stop.

Step 5.
$$\begin{array}{r} x + 2 \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \end{array}$$

Ask yourself "What must I multiply x by to get exactly $2x$?" ($2x$ is the first term of the new dividend.) The answer is 2. This is placed in the quotient position.

Step 6.
$$\begin{array}{r} x + 2 \\ x + 3\overline{)x^2 + 5x + 6} \\ \underline{x^2 + 3x} \\ 2x + 6 \\ \underline{2x + 6} \\ 0 \end{array}$$

Repeat steps 3 through 5 as often as necessary until you answer the question in step 4 with *YES*.

Example 1

Divide $3x^3 - 2x - 11x^2 + 24$ by $3x + 4$.

Solution

Always arrange the terms of both the divisor and the dividend in descending powers of one of the variables. This problem rewritten ready for long division is

$$3x + 4\overline{)3x^3 - 11x^2 - 2x + 24}$$

Now, performing the division we have

$$\begin{array}{r} x^2 - 5x + 6 \\ 3x + 4\overline{)3x^3 - 11x - 2x + 24} \\ \underline{3x^3 + 4x^2} \\ -15x^2 - 2x + 24 \\ \underline{-15x^2 - 20x} \\ 18x + 24 \\ \underline{18x + 24} \\ 0 \end{array}$$

Check $(x^2 - 5x + 6)(3x + 4) = \ ?$

Example 2

Divide $-5x^2 - 2x^3 + 3$ by $3 + x$.

Solution

First, arrange the terms in descending powers of the variable x.

$$x + 3\overline{)-2x^3 - 5x^2 + 3}$$

Whenever there is a term left out in the descending power statement indicate this by placing a zero in its place. In this case we will use $0x$ to indicate there is no term containing x. Now we are ready to carry out the division.

$$\begin{array}{r} -2x^2 + x - 3 \\ x + 3\overline{)-2x^3 - 5x^2 + 0x + 3} \\ \underline{-2x^3 - 6x^2} \\ x^2 + 0x + 3 \\ \underline{x^2 + 3x} \\ -3x + 3 \\ \underline{-3x - 9} \\ 12 \end{array}$$

Notice there is a remainder. Remember how to check an ordinary division of natural numbers when there is a remainder?

$$\begin{array}{r} 4 \\ 14\overline{)68} \\ \underline{56} \\ 12 \end{array}$$

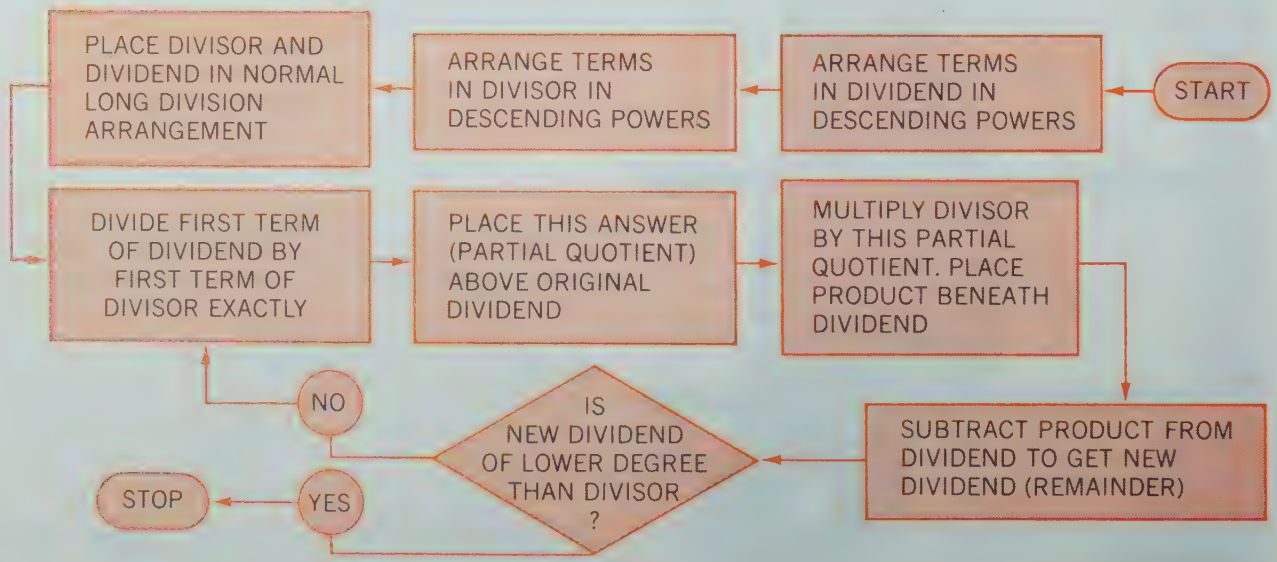
Dividend = divisor × quotient + remainder

Check $(14 \times 4) + 12 = 68$

To check in the example we have:

$$(-2x^2 + x - 3)(x + 3) + 12 = -2x^3 - 5x^2 - 9 + 12$$
$$= -2x^3 - 5x^2 + 3 \checkmark \text{ Checks.}$$

This process of long division is shown in flow chart form.


START
ARRANGE TERMS IN DIVIDEND IN DESCENDING POWERS
ARRANGE TERMS IN DIVISOR IN DESCENDING POWERS
PLACE DIVISOR AND DIVIDEND IN NORMAL LONG DIVISION ARRANGEMENT
DIVIDE FIRST TERM OF DIVIDEND BY FIRST TERM OF DIVISOR EXACTLY
PLACE THIS ANSWER (PARTIAL QUOTIENT) ABOVE ORIGINAL DIVIDEND
MULTIPLY DIVISOR BY THIS PARTIAL QUOTIENT. PLACE PRODUCT BENEATH DIVIDEND
SUBTRACT PRODUCT FROM DIVIDEND TO GET NEW DIVIDEND (REMAINDER)
IS NEW DIVIDEND OF LOWER DEGREE THAN DIVISOR ?
NO
YES
STOP

EXERCISE 5·12

Write each numeral in expanded form and divide by long division. The first is written in expanded form.

1. $40 + 2\overline{)800 + 80 + 2}$
2. $21\overline{)693}$
3. $41\overline{)861}$
4. $32\overline{)352}$
5. $4\overline{)484}$
6. $3\overline{)693}$

Find the quotients by long division. Refer to the examples and to the flow chart when necessary. Check each.

7. $\frac{30x^2 + 15}{5}$
8. $\frac{28a^2b - 14ab^2}{7a}$
9. $\frac{42a^2 + 12a}{3a}$
10. $\frac{56x^2 - 4x^3}{4x^2}$
11. $\frac{72y^2 - 81y}{9y}$
12. $\frac{65c^2 - 91c^4}{13c^2}$
13. $3b\overline{)27b^3 - 81b^2 + 6b}$
14. $ab^2\overline{)30a^3b^4 + 5a^2b^3 - 4ab^2}$
15. $\frac{4x^2 + 14x + 6}{x + 3}$
16. $\frac{3x^2 - 4x - 4}{x - 2}$
17. $x - 2\overline{)-7x^2 + 20x - 12}$
18. $2x + 1\overline{)124x^2 + 50x - 6}$
19. $(-8x + x^3 - 32) \div (x - 4)$
20. $(2x^3 - x^2 - 24x - 16) \div (x - 4)$
21. $(a^2 + 5a + 4) \div (a + 2)$
22. $(x^3 - 8) \div (x - 2)$
23. $(m^2 - 9) \div (m - 3)$
24. $(3 - 2n + 3n^2) \div (n - 2)$
25. $(x^3 - 27) \div (x - 3)$
26. $(2x - x^3 - 3) \div (4 + 2x)$
27. $(6x^4 - x - 2x^2 + 1) \div (3x + 2)$
28. $(3 - 4x^2 - x + 4x^3) \div (2x - 2)$
29. Write a flow chart for checking question 28 above. Compare your flow chart with your neighbour's.
30. $(x^2 + 18x + 81) \div (x + 9)$
31. $(x^2 - 81) \div (x - 9)$
32. $(x^2 + 14x + 49) \div (x + 7)$
33. $(x^2 - 49) \div (x - 7)$
34. $(x^2 + 12x + 36) \div (x + 6)$
35. $(x^2 - 36) \div (x - 6)$
36. $(x^2 + 8x + 16) \div (x + 4)$
37. $(x^2 - 16) \div (x - 4)$
38. $(x^2 + 10x + 25) \div (x + 5)$
39. $(x^2 - 25) \div (x - 5)$
40. Following the pattern established in questions 30 through 39, write the following polynomial in its factored form: $x^2 - 64$.

5·13 Division With Remainder Zero

In the preceding section we saw two types of division of polynomials. In one case the remainder was zero while in the other case the remainder was some term other than zero. We can summarize the two cases:

1. If a is divided by b and the remainder is zero, then $a = b \cdot c$ where a, b, c are polynomials.
2. If a is divided by b and the remainder is other than zero, then $a = b \cdot c + r$ where a, b, c, r are polynomials.

The case in which the remainder is zero is one of importance to us. When a polynomial a is written as a product of two other polynomials b and c with no remainder, then we know b and c are factors of a. We will illustrate this with two instances using natural numbers then with an instance using polynomials in one variable.

$15 = 3 \times 5$. Therefore, 3 and 5 are factors of 15.
$46 = 2 \times 23$. Therefore, 2 and 23 are factors of 46.
$a^2 + 2a - 3 = (a - 1)(a + 3)$. Therefore $(a - 1)$ and $(a + 3)$ are factors of $a^2 + 2a - 3$.

On the other hand consider:

$$\begin{array}{r} 3 \\ 4\overline{)14} \\ \underline{12} \\ 2 \text{ remainder} \end{array}$$

Hence we know 4 is not a factor of 14.

$$\begin{array}{r} a + 2 \\ a + 1\overline{)a^2 + 3a + 5} \\ \underline{a^2 + a} \\ 2a + 5 \\ \underline{2a + 2} \\ 3 \text{ remainder} \end{array}$$

Hence we know $a + 1$ is not a factor of $a^2 + 3a + 5$.

Example 1

Determine whether or not $2a - 3$ is a factor of $6a^2 + 5a - 3$.

Solution

Use long division to divide $6a^2 + 5a - 3$ by $2a - 3$.

$$\begin{array}{r} 3a + 7 \\ 2a - 3\overline{)6a^2 + 5a - 3} \\ \underline{6a^2 - 9a} \\ 14a - 3 \\ \underline{14a - 21} \\ 18 \text{ remainder} \end{array}$$

Since there is a remainder $2a - 3$ is not a factor of $6a^2 + 5a - 3$.

Example 2

Determine whether or not $2a - 3$ is a factor of

$$4a^2 - 16a + 15$$

Solution

Divide

$$\begin{array}{r} 2a - 5 \\ 2a - 3\overline{)4a^2 - 16a + 15} \\ \underline{4a^2 - 6a} \\ -10a + 15 \\ \underline{-10a + 15} \\ 0 \end{array}$$

Since the remainder is zero (we usually say there is no remainder) we know $2a - 3$ is a factor.
Another factor is $2a - 5$.

EXERCISE 5·13

Determine whether or not the monomial or binomial on the right is a factor of the polynomial on the left. If it is, find the second factor.

1.	756	3
2.	9345	9
3.	$x^2 + 2x + 5$	$x + 1$
4.	$m^2 + m - 2$	$m + 2$
5.	$3y^2 - 3y + 3$	$3y + 1$
6.	$3y^2 - 3y + 3$	$y + 1$
7.	$6n^2 - 6n + 2$	$2n + 1$
8.	$3z^3 - 2z + 4$	$z + 2$
9.	$3w^2 - 2w + 1$	$w - 1$
10.	$14s^2 - 17s - 6$	$2s - 3$
11.	$x^3 + 3x^2 + 3x - 1$	$x + 1$

12. Find the value of k such that $8x^2 + 2x - k$ is divisible by $4x + 3$. *Hint!* Use long division and make the replacement for k in the last step of the division such that there is no remainder.

13. Find the value of k such that $6x^2 - 7x - k$ is divisible by $3x - 4$.

Magic squares have captured the interest of many people. Can you find Benjamin Franklin's magic square in the library.

5·14 Solution of Equations in One Variable in R

This section is intended to extend the work you did in Chapter 2 on the solution of equations. You will be required to use the skills you have learned in multiplying polynomials. The questions will involve the set of real numbers.

Example 1

Solve $\dfrac{(x-3)(x+1)}{3} = \dfrac{(x-1)(x+2)}{3}$

Solution

$$\frac{(x-3)(x+1)}{3} = \frac{(x-1)(x+2)}{3}$$

$$\frac{3(x-3)(x+1)}{3} = \frac{3(x-1)(x+2)}{3}$$ {Multiply each side by LCM 3

$$x^2 - 2x - 3 = x^2 + x - 2$$

$$x^2 - x^2 - 2x - 3 = x^2 - x^2 + x - 2$$ {Subtract x^2 from each side

$$-2x - 3 = x - 2$$

$$-2x - x - 3 = x - x - 2$$ {Subtract x from each side

$$-3x - 3 = -2$$

$$-3x - 3 + 3 = -2 + 3$$ {Add 3 to each side

$$-3x = +1$$

$$\frac{-3x}{-3} = \frac{1}{-3}$$ {Divide each side by −3

$$x = -\tfrac{1}{3}$$

EXERCISE 5·14

Solve for x in each and verify the odd numbered questions.

1. $(x+1)(x+4) = (x+3)(x-5)$
2. $(x+2)(x+1) = (x+2)(x-5)$
3. $(x-2)(x+1) = (x+1)^2$
4. $(x-1)(x+1) = (x+4)^2$
5. $(x+1)(x-6) = (x-7)^2$
6. $x - \pi = 2x - 2\pi$
7. $\dfrac{2x-3}{3} + \dfrac{3x+2}{4} = \dfrac{4x-1}{3} - (x-2)$
8. The length of a rectangle is 8 inches longer than its width. The length of a second rectangle is 10 inches longer than the width of the first. The second rectangle is 1 inch narrower than the width of the first. The two rectangles have the same area. Find the dimensions of the two rectangles.
9. The top base of a trapezoid is 2 inches shorter than the bottom base. The altitude is equal to the bottom base. The area of the trapezoid is 17 square inches less than the square of the base. Find the dimensions and the area of the trapezoid.
10. The height of a cylindrical can is 2 inches more than the measure of the circumference. The lateral area of the can is 8 inches more than the square of the circumference. Find the circumference, the height, and the lateral area of the can. *Hint!* Let x represent the circumference.
11. Construct a flow chart for the solution of question 1.

5·15 Absolute Value

Absolute value can be defined with reference to a number line.

We have two points exactly the same distance from point 0. This distance from the reference point 0 is sometimes referred to as *absolute value*.

The point labelled +5 is 5 units from the point 0.
The point labelled −5 is 5 units from the point 0.

What two points are 7 units from point 0? What two numbers have absolute value 7?

What is the absolute value of +10? −10? 14? −14?
Do question 1 in Exercise 5·15.

Consider the absolute value of x. This is written $|x|$.

If $x = +9$ then $|x| = |+9|$
$= 9.$

If $x = -9$ then $|x| = |-9|$
$= 9.$

If $x = 0$ then $|x| = |0|$
$= 0.$

Example 1

Find $|x_1 - x|$ given

(a) $x_1 = 7, \quad x = 9$

(b) $x_1 = 16, x = 5$

(c) $x_1 = 23, x = 23.$

Solution

(a) $|x_1 - x| = |7 - 9|$
$= |-2|$
$= 2$ $\quad -2$ is two units from 0.

(b) $|x_1 - x| = |16 - 5|$
$= |11|$
$= 11$ $\quad 11$ is eleven units from 0.

(c) $|x_1 - x| = |23 - 23|$
$= |0|$
$= 0$ $\quad 0$ is zero units from 0.

Students in electricity and physics use absolute value. Absolute value is used later in this book in the chapter on vectors.

EXERCISE 5·15

1. Give the absolute value of each number.

   (a) 41 (b) $-\frac{3}{4}$ (c) $-4\frac{1}{5}$ (d) 0
   (e) 49 (f) $31\frac{1}{3}$ (g) $5\frac{1}{4}$ (h) -23

2. Give $|x_1 - x|$ when:

   (a) $x_1 = 13, x = -4$ (b) $x_1 = 13, x = 19$
   (c) $x_1 = 6, x = 22$ (d) $x_1 = -32, x = 32$
   (e) $x_1 = -7, x = -23$ (f) $x_1 = -33, x = -7$

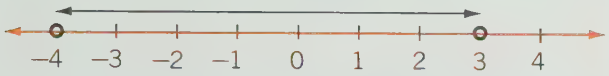
3. Write $|x - x_1|$ given the values of x_1 and x shown in question 2. What did you notice about these answers compared to the answers in question 2?

4. Write $|x - x_1|$ when (x_1, x) is:

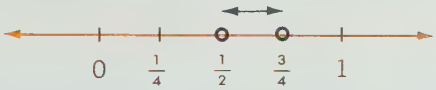
   (a) (18,15) (b) (−4,−11)
   (c) (−32,−32) (d) (−8,−10)
   (e) (13,−23) (f) $(-4\frac{1}{3}, -11\frac{5}{6})$
   (g) $(41, 31\frac{3}{4})$ (h) $(5\frac{3}{4}, -6\frac{1}{8})$

5·16 Which Is Larger?

If you were asked which is the larger number: 3 or −4, you would answer "3 is larger since it appears to the right of −4 on the number line."



If you were asked which is the larger number: $\frac{3}{4}$ or $\frac{1}{2}$, again you would answer "$\frac{3}{4}$ is larger since it appears to the right of $\frac{1}{2}$ on the number line."



What would you answer if you were asked which is the larger number: $\frac{6}{57}$ or $\frac{20}{193}$? How would you attempt to solve this problem since it would be very difficult to show these two numbers on a number line?

Do not read further
until
you have attempted the question.

One method is to change each to an equivalent decimal numeral and to compare the two. Change each to a decimal numeral correct to 3 decimal places.

$$\frac{6}{57} = 0.\cdots \qquad \frac{20}{193} = 0.\cdots$$

Which of the two numbers is the larger?

Now use this method to find which is the larger:

$$\frac{3}{1011} \text{ or } \frac{5}{1733}?$$

Did you find it quite a bit of work to change each fraction to an equivalent decimal numeral (the slide rule is not accurate enough)? Have you wondered if there is an easier way?

Another way to determine which is the larger is shown here.

(a) $\frac{6}{57} \times \frac{193}{193} = \frac{1158}{10901}$ (b) $\frac{20}{193} \times \frac{57}{57} = \frac{1140}{10901}$

$$\frac{1158}{10901} > \frac{1140}{10901}$$

$$\therefore \quad \frac{6}{57} > \frac{20}{193}$$

Which is larger?

You noticed that we expressed the two fractions with the equal denominators. Since the denominators are equal we can, in effect, ignore the denominators. We are only interested in the new numerators.

We found the new numerator in (a) by multiplying the numerator, 6, by the denominator in the second fraction, 193.

$$\frac{6}{57} \quad \frac{20}{193}$$

Similarly, we found the new numerator in (b) by multiplying the numerator 20, by the denominator in the first fraction, 57.

$$\frac{6}{57} \quad \frac{20}{193}$$

Since $6 \times 193 > 20 \times 57$ then $\frac{6}{57} > \frac{20}{193}$.

Example 1

Express each pair of fractions as fractions with the same denominator. Which fraction is the larger number?

(a) $(\frac{2}{3}, \frac{5}{9})$ (b) $(\frac{7}{15}, \frac{8}{17})$

Solution

(a) $\frac{2}{3} \times \frac{9}{9} = \frac{2 \times 9}{3 \times 9} = \frac{18}{27}$ $\frac{5}{9} \times \frac{3}{3} = \frac{5 \times 3}{9 \times 3} = \frac{15}{27}$

Since $\frac{18}{27} > \frac{15}{27}$ then $\frac{2}{3} > \frac{5}{9}$.

(b) $\frac{7}{15} \times \frac{17}{17} = \frac{119}{15 \times 17}$ $\frac{8}{17} \times \frac{15}{15} = \frac{120}{17 \times 15}$

Since $120 > 119$ then $\frac{8}{17} > \frac{7}{15}$.

Since we need be concerned only with the numerators we will write only the numerator. Study the charts below. Each statement on the left is obvious. Complete each statement on the right by inserting the appropriate symbol ($>$, $=$, $<$) or the appropriate numbers.

$\frac{1}{2} > \frac{1}{4}$	$1 \times 4 > 2 \times 1$
$\frac{7}{8} > \frac{1}{2}$	$7 \times 2 \;?\; 8 \times 1$
$\frac{5}{8} > \frac{1}{4}$	$5 \times 4 \;?\; 8 \times 1$
$\frac{9}{10} > \frac{2}{5}$	$9 \times 5 \;?\; 10 \times 2$
$\frac{7}{9} > \frac{1}{3}$	$? \times ? > ? \times ?$
$\frac{4}{5} > \frac{1}{2}$	$? \times ? > ? \times ?$
$\frac{7}{8} > \frac{2}{4}$	$? \times ? > ? \times ?$
$\frac{a}{b} > \frac{c}{d}$	$? \times ? > ? \times ? \; \frac{a}{b}, \frac{c}{d} > 0$

Now complete this section in a similar manner.

Given the two fractions

$\frac{1}{2}, \frac{1}{4}$ if $1 \times 4 > 2 \times 1$ then $\frac{1}{2} > \frac{1}{4}$

$\frac{7}{8}, \frac{1}{2}$ if $7 \times 2 > 8 \times 1$ then $\frac{7}{8} > \frac{1}{2}$

$\frac{5}{8}, \frac{1}{4}$ if $5 \times 4 > 8 \times 1$ then $? > ?$

$\frac{3}{4}, \frac{1}{8}$ if $3 \times 8 > 4 \times 1$ then $? > ?$

$\frac{5}{6}, \frac{2}{3}$ if $5 \times 3 > 6 \times 2$ then $? > ?$

$\frac{7}{9}, \frac{2}{7}$ if $? > ?$ then $? > ?$

$\frac{8}{11}, \frac{3}{5}$ if $? > ?$ then $? > ?$

$\frac{a}{b}, \frac{c}{d}$	if $? > ?$ then $? > ?$	$\frac{a}{b}, \frac{c}{d} > 0$
$\frac{1}{5}, \frac{2}{3}$	if $1 \times 3 < 5 \times 2$ then $? < ?$	
$\frac{4}{7}, \frac{7}{9}$	if $4 \times 9 < 7 \times 7$ then $? < ?$	
$\frac{3}{11}, \frac{7}{8}$	if $? < ?$ then $? < ?$	
$\frac{4}{15}, \frac{8}{11}$	if $? < ?$ then $? < ?$	
$\frac{a}{b}, \frac{c}{d}$	if $? < ?$ then $? < ?$	$\frac{a}{b}, \frac{c}{d} > 0$

EXERCISE 5·16

Determine which relation ($>$, $=$ or $<$) exists between the first and second component of each pair. Do the first 6 orally.

1. $(\frac{3}{4}, \frac{11}{15})$
2. $(\frac{3}{8}, \frac{10}{25})$
3. $(\frac{2}{3}, \frac{13}{19})$
4. $(\frac{8}{9}, \frac{10}{11})$
5. $(\frac{10}{12}, \frac{5}{6})$
6. $(\frac{7}{21}, \frac{3}{16})$
7. $(\frac{13}{18}, \frac{39}{55})$
8. $(\frac{12}{23}, \frac{7}{13})$
9. $(\frac{12}{89}, \frac{7}{50})$
10. $(\frac{31}{65}, \frac{41}{75})$
11. $(\frac{13}{17}, \frac{23}{27})$
12. $(\frac{24}{30}, \frac{60}{75})$

13. Try to devise a simple rule to decide which of two fractions is the larger, when one fraction is negative and the other is positive.

14. Try to devise a simple rule to decide which of two fractions is the larger when both fractions are less than zero (negative).

15. Apply your rule to decide which relation, ($>$, $=$, or $<$) exists between the first and second component of each pair.

    (a) $(-\frac{1}{4}, -\frac{1}{2})$ (b) $(-\frac{3}{8}, -\frac{7}{19})$ (c) $(-\frac{1}{2}, \frac{1}{2})$

    (d) $(-\frac{13}{23}, -\frac{4}{7})$ (e) $(-\frac{3}{2}, \frac{1}{8})$ (f) $(\frac{23}{39}, -\frac{70}{75})$

Determine the proper relation ($>$, $=$, $<$) between the first and second component of each pair after the indicated operation.

16. $(\frac{2}{3}, \frac{3}{4})$ after
    (a) adding $\frac{1}{3}$ to each
    (b) adding $-\frac{1}{3}$ to each
    (c) multiplying each by $\frac{2}{3}$
    (d) multiplying each by $\frac{3}{4}$
    (e) dividing each by $\frac{2}{3}$
    (f) dividing each by $-\frac{3}{4}$

17. $(\frac{7}{8}, \frac{5}{6})$ after
    (a) adding $\frac{3}{5}$ to each
    (b) adding $-\frac{3}{5}$ to each
    (c) multiplying each by $\frac{3}{5}$
    (d) multiplying each by $-\frac{6}{7}$
    (e) dividing each by $\frac{3}{5}$
    (f) dividing each by $-\frac{6}{7}$

5·17 Solutions of Inequalities

The rules that arise from the exercises in the previous section are listed here in general terms.

Rule 1

> If a number is added or subtracted from both sides of an inequality, the statement of inequality remains true.

Rule 2

> If both sides of a statement of inequality are multiplied or divided by a *positive* number, the statement of inequality remains true.

Rule 3

> If both sides of a statement of inequality are multiplied or divided by a *negative* number, the sense of the inequality is reversed. (Negative multiplier property of inequalities.)

Let us illustrate Rules 2 and 3.

Given:	$8 > 2$
then	$8 \times 4 \;?\; 2 \times 4$
	$32 \;?\; 8$
Obviously	$32 > 8.$
But consider:	$8 > 2$
then	$8 \times (-4) \;?\; 2 \times (-4)$
	$-32 \;?\; -8$
Obviously	$-32 < -8.$

Notice that when each side of the inequality was multiplied by a number less than zero the sense of the inequality changed.

Now complete these two:

$$8 > 4 \qquad\qquad 8 > 4$$
$$8 \div 2 \;?\; 4 \div 2 \qquad\qquad 8 \div (-2) \;?\; 4 \div (-2)$$

Example 1

Solve for x and graph the solution set.

$$2x + 3 < 15$$

Solution

$$2x + 3 < 15$$
$$2x + 3 - 3 < 15 - 3 \qquad \text{Subtract 3 from each side.}$$
$$2x < 12$$
$$\frac{2x}{2} < \frac{12}{2} \qquad \text{Divide each side by 2.}$$
$$x < 6$$
$$\{x \mid x < 6\}$$

Example 2

Solve for x and graph the solution set.

$$\left\{x \left| \frac{3-4x}{6} > \frac{16+4x}{3} \right.\right\}$$

Solution

$$\frac{3-4x}{6} > \frac{16+4x}{3}$$

$6\left(\frac{3-4x}{6}\right) > 6\left(\frac{16+4x}{3}\right)$ {Multiply each side by L.C.M.

$3-4x > 2(16+4x)$

$3-4x > 32+8x$

$3-3-4x > 32-3+8x$ Subtract 3 from each side.

$-4x > 29+8x$

$-4x-8x > 29+8x-8x$ {Subtact $8x$ from each side.

$-12x > 29$

$\frac{-12x}{-12} < \frac{29}{-12}$ {Divide each side by -12. (Reverse sense of the inequality.)

$x < -\frac{29}{12}$

$\{x|x < -\frac{29}{12}\}$

Example 3

Solve for x. Graph the solution set. (Optional)

$\{x|(x-1)(x+2) \geq x^2+8 \text{ or } (x-3)(x+2) > (x-1)(x+1)\}$

Solution

Work with each inequality in turn.

$(x-1)(x+2) \geq x^2+8$

$x^2+x-2 \geq x^2+8$ Product of two binomials.

$x^2-x^2+x-2 \geq x^2-x^2+8$ Subtract x^2 from each.

$x-2 \geq 8$

$x-2+2 \geq 8+2$ Add 2 to each side.

$x \geq 10$

The second inequality:

$(x-3)(x+2) > (x-1)(x+1)$

$x^2-x-6 > x^2-1$ Products of binomials

$x^2-x^2-x-6 > x^2-x^2-1$ {Subtract x^2 from each side.

$-x-6 > -1$

$-x-6+6 > -1+6$ Add 6 to each side.

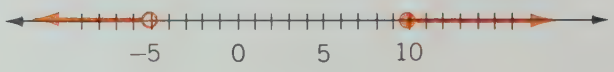
$-x > 5$

$-1(-x) < -1(5)$ {Multiply each side by -1. Reverse sense of the inequality.

$x < -5$

Combining the two solution sets:

$\{x|x \geq 10 \quad \text{or} \quad x < -5\}$



EXERCISE 5·17

Solve for x in each. Graph each solution set.

1. $x-1 \leq 6$
2. $3x+5 < 11$
3. $5-2x < 12$
4. $-22x > 110$
5. $22-3x < 6$
6. $5x-13 > 16x+4$
7. $6x+3 \geq 14x-6$
8. $\frac{3}{2}x-\frac{2}{3} \geq \frac{13}{3}x-\frac{3}{4}$
9. $\frac{7}{8}+\frac{3}{4}x \leq \frac{8}{3}x-3$
10. $\frac{x+3}{2}+\frac{x-2}{6} \geq \frac{2x-3}{3}$

Find the solution set for each.

11. $(x-2)(x-4) > x^2+18$
12. $(x-3)(x-5) < x^2-6$
13. $(x+4)(x+2) < (x-2)(x-3)$
14. $(x-5)(x+3) \geq (x+6)(x-9)$
15. $2(x-1)(x+1) \leq 2(x-2)(x+3)$

Find the solution set for each. Graph the solution set (optional).

16. $\{x|(x+3)(x-2) > x^2-6 \text{ and } x-1 < -4\}$
17. $\{x|x^2-3 \geq (x-4)(x-5) \text{ and } 1-3x > 4\}$

Using the Library

1. *The Giant Golden Book of Mathematics*. Read page 26. Find the value of π by the method described (instead of planks on the floor use a sheet of paper with lines that are apart the length of the nails picks).

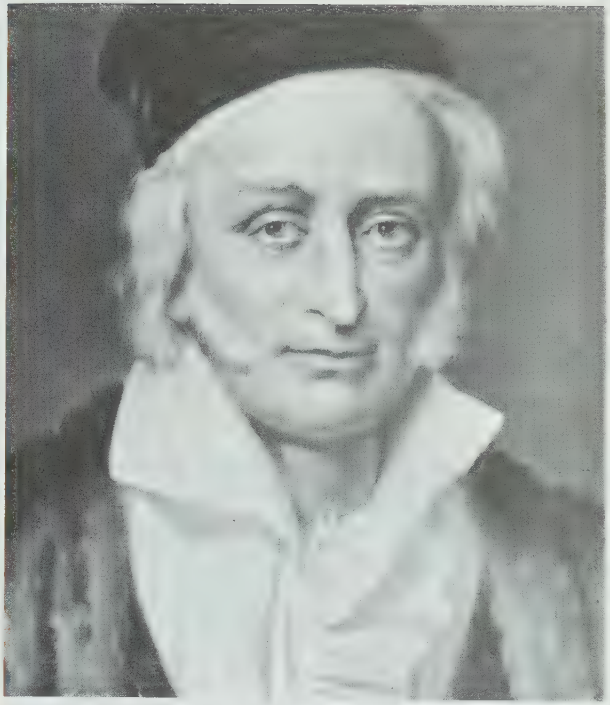
   Also read page 66. Find values for π by using the two formulas on the bottom of page 66.

   Read pages 39-45. Account for the necessity of different numbers for the shepherd, the bookkeeper, the carpenter, and the electrician. Give an example of the numbers each must use.

   Read pages 88-89. What contributions did Gauss make to astronomy?

2. *Mathematics Enrichment*. Read pages 68 to 74. Make the apparatus as shown on page 73 to find the value of π on a number line.

3. *The Mathmen*. Read pages 77 and 97-98. Explain Eudoxus' theory of the movement of heavenly bodies.



Karl Friedrich Gauss (1777-1855). Probably the greatest mathematician who has ever lived.

Is it possible that these people could use math?

REVIEW EXERCISE 5

Make the indicated calculations.

1. $-\frac{3}{7} \times \frac{2}{9}$
2. $\frac{18}{5} \div \frac{3}{20}$
3. $\frac{3}{5} + \frac{1}{3}$
4. $\frac{5}{8} - \frac{8}{5}$
5. $-\frac{3}{7} - \left(-\frac{2}{21}\right)$
6. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{5}{8}}$

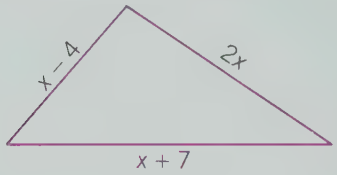
Round to the nearest hundredth.

7. 4.135
8. 0.5655
9. 0.485
10. 41.12954

Solve and verify each.

11. $3x - 12 = 13$
12. $\frac{3}{5}(x + 2) = 3x - 2$

13. The length of the sides of a triangle are shown in the diagram. If the perimeter of the triangle is 57 inches how long is each side?



14. The total resistance R(in ohms) of two parallel resistances, r_1 and r_2 (in ohms) is given by the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

Find

(a) R given $r_1 = 12$, $r_2 = 8$

(b) r_1 given $R = \frac{24}{7}$, $r_2 = 8$

15. Evaluate:

(a) $3(5 - 2) \div 4[2(6 + 3)]$

(b) $14.7d - 15.6d + 33.1d$

16. Multiply:

(a) $(3a + 4d)(6a + 2d)$ (b) $(2b - 3d)(2b + 3d)$

(c) $(m - 2n)^2$

17. Factor:

(a) $ax + ay + az$ (b) $\frac{3}{5}t^2 + \frac{3}{5}t$

(c) $2x(y - 3z) + 6(y - 3z)$ (d) $4a^2 - 9b^2$

18. Find the quotients by long division.

(a) $3a\overline{)6a^2 - 12a}$ (b) $3a - 6\overline{)6a^2 - 9a - 6}$

19. Is $x - 2$ a factor of

(a) $x^2 - 4x + 4$ (b) $2x^2 - x - 4$?

20. Find $|x - y|$ given $x = 3$, $y = 8$.

21. Which is larger: $\frac{3}{13}$ or $\frac{7}{29}$?

22. Solve for x: $(x - 2)(x - 4) > x^2 + 6$.

For various reasons, it is necessary for us to know our inventory on a monthly basis. With over 7,000 items in stock, it is not practical to count and cost each item every month so we have to calculate our month-end inventory by the gross profit method.

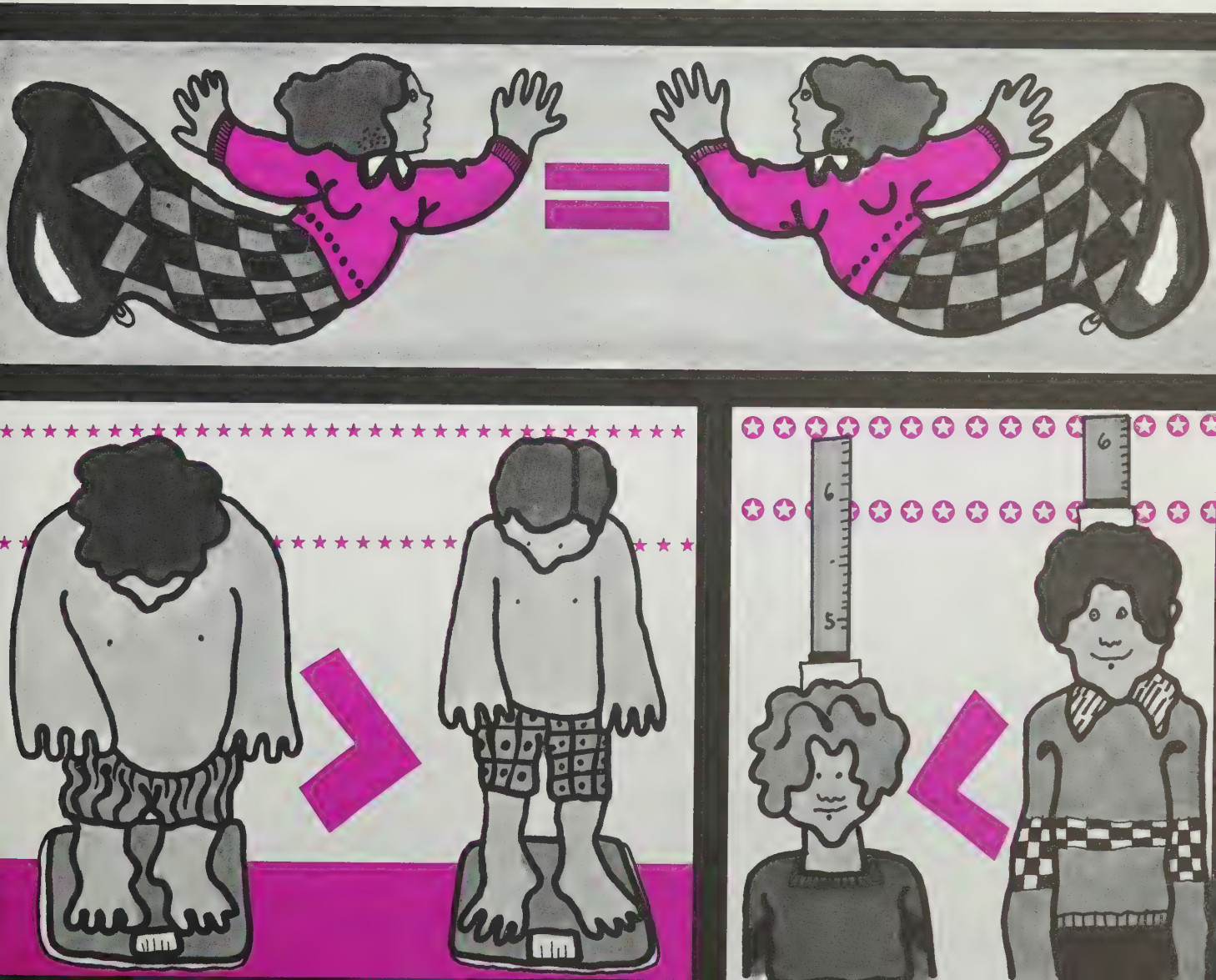
Problem 1—On financial statements we calculate all costs as a percentage of sales. Calculate to 2 decimal places.

Sales	$175,000	100.00%
Administrative Costs:		
Salaries	22,360	
Rent	2,500	
Insurance	575	
Utilities	210	
Auto	265	
Depreciation	300	
Bad Debts	500	
Collection Expense	50	
Taxes	200	
Repairs & Maintenance	766	
Supplies	135	
Advertising	1,810	
Miscellaneous	96	
Total	29,767	

Problem 2—Prorate the following costs of Rent $2,500, Insurance $575, and Taxes $200, to the following departments on the basis of square footage occupied by each department.

Stationary	-6,000 sq. ft.
Furniture	-3,575 sq. ft.
Printing	-766 sq. ft.
Systems	-356 sq. ft.
Shipping	-1,230 sq. ft.
Receiving	-475 sq. ft.
Warehouse	-6,000 sq. ft.

Chapter Six
LINEAR EQUALITIES AND INEQUALITIES



6·1 Graphical Solutions of a System of Equations

The graph of $\{(x,y)|y = 2x - 4\}$ is an infinite set of points lying on a straight line in the Cartesian plane. The graph of $\{(x,y)|y = -x + 2\}$ also is a straight line. The intersection of these two sets is the set of points common to both lines. In this case there is only one point in common. This point (2,0) is the only element of the solution set. The two equations, $y = 2x - 4$ and $y = -x + 2$, are called a *system of equations*. The ordered pair (2,0) is the *solution* to the system.

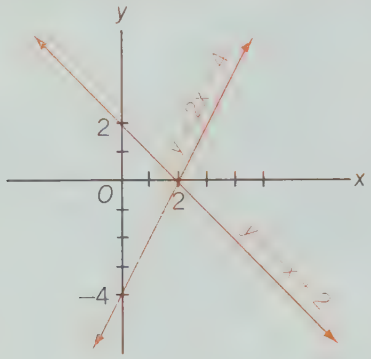


Figure 6-1

We may use two different ways of stating the problem. Essentially we are asking the same thing when we write:

(a) Solve: $\{(x,y)|y = 2x - 4\} \cap \{(x,y)|y = -x + 2\}$

(b) Solve: $\begin{cases} y = 2x - 4 \\ y = -x + 2. \end{cases}$

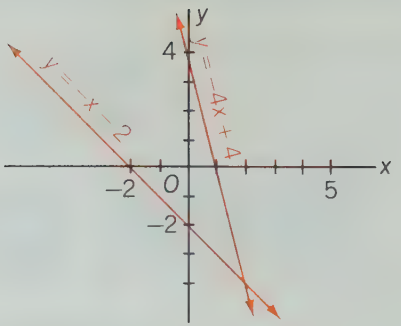
Example 1

Solve graphically:

$$\{(x,y)|y = -4x + 4\} \cap \{(x,y)|y = -x - 2\}$$

Solution

We graph the two equations as shown in the diagram below. The two lines intersect at one point. The two lines have the point (2,−4) in common.



Check by making the replacements in each of the original equations.

$$\begin{aligned} y &= -4x + 4 \\ -4 \;&\text{?}\; -4(2) + 4 \\ -4 &= -4 \\ y &= -x - 2 \\ -4 \;&\text{?}\; -(2) - 2 \\ -4 &= -4 \end{aligned}$$

The solution is (2,−4) and the solution set is $\{(2,-4)\}$. Such a system which has a unique solution is called *consistent*.

> A system is said to be consistent when the two equations represent intersecting lines.

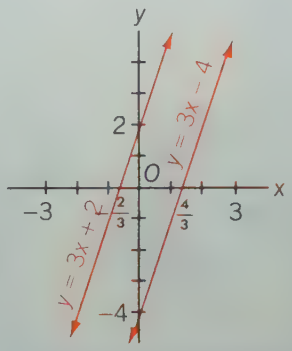
Example 2

Solve graphically:

$$\begin{cases} y = 3x + 2 \\ y = 3x - 4 \end{cases}$$

Solution

We graph the two equations.



Notice the two lines are parallel. The two lines have no point in common. There is no solution for these two equations. We say the solution set is the empty set: $\varnothing$ or $\{\ \}$. Such a system which has no solution set is called *inconsistent.*

> A system is said to be inconsistent when the equations represent two distinct parallel lines.

Example 3

Rod is a star hockey player. Four times the number of minutes Rod spent in the penalty box plus six times the number of minutes on the ice total 48 minutes. Also it is known that eight times the number of minutes Rod spent in the penalty box plus twelve times the number of minutes on the ice total 96. How many minutes did he spend in the penalty box and how many on the ice?

Solution

We must write the equations first.

Let x represent the number of minutes in the penalty box.

Let y represent the number of minutes on the ice.

4 times min. in box + 6 times min. on ice = 48

(1) $4 \times x + 6 \times y = 48$

8 times min. in box +12 times min. on ice = 96

(2) $8 \times x + 12 \times y = 96$

(1) $4x + 6y = 48$
(2) $8x + 12y = 96$

Graph the two equations:

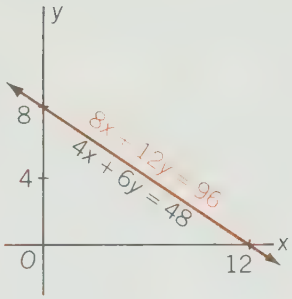
Use the *two-point* method.

(1)

x	0	12	3
y	8	0	6

(2)

x	0	12	3
y	8	0	6



Both equations produce the same line. Since the two lines have many points in common, there are many solutions to the problem. In fact, there is an *infinite* number of solutions. From the graph we see that (0,8), (12,0), (3,6) are but three solutions. Each point on the line is a solution. Such a system, whose graphs have all points in common, is called *dependent.*

> A system is said to be dependent when the equations represent the same line.

If we are given an equation such as

(1) $x + y = 5,$

can we write other equations that have the same solution set? Let us look at some of the elements of the solution set of equation (1):

x	−3	−2	−1	0	1	2	3	4	5	6	7
y	8	7	6	5	4	3	2	1	0	−1	−2

Now let us multiply the whole equation by 2.

$$2(x + y = 5)$$

(2) $$2x + 2y = 10$$

Some of the elements of the solution set of this new equation are:

x	−3	−2	−1	0	1	2	3	4	5	6	7
y	8	7	6	5	4	3	2	1	0	−1	−2

Apparently the second equation has the same solution set as the first. We say the two equations are *equivalent.*

Let us multiply equation (1) by 3.

$$3(x + y = 5)$$

(3) $$3x + 3y = 15$$

Some of the elements of the solution set of equation (3) are:

x	−3	−2	−1	0	1	2	3	4	5	6	7
y	8	7	6	5	4	3	2	1	0	−1	−2

Equation (3) is equivalent to the other two equations.

Now we could multiply equation (1) by any number and we would obtain an equation equivalent to the others. Let us try one more.

Multiply equation (1) by − 6.

$$-6(x + y = 5)$$

(4) $$-6x - 6y = -30$$

Part of the solution set of equation (4) is:

x	−3	−2	−1	0	1	2	3	4	5	6	7
y	8	7	6	5	4	3	2	1	0	−1	−2

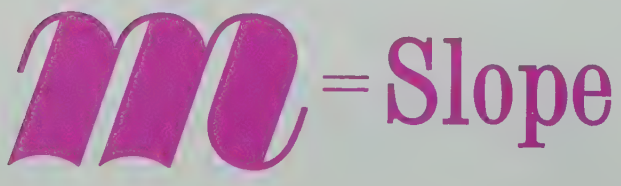
Two or more equations are equivalent equations if each has the same solution set.

The linear equation $x + y = b$ is equivalent to $a(x + y = b)$ or $ax + ay = ab$.

EXERCISE 6·1

1. If two lines intersect, how many points can the two lines have in common?

2. Compare the slopes of the two lines in the graph of Example 1. Are the slopes equal? Now look at the equations in Example 1. Are the m values equal in the two equations?



3. Compare the slopes of the two lines in the graph of Example 2. Are the m values equal in the two equations? Are the b values equal?

4. Refer to Example 3. Transform each equation to the $y = mx + b$ form. Are the m values of each equation equal? Are the b values equal? Now look at the original two equations. Compare the ratios of the coefficients of y, x, and the ratio of the b's. What do you notice? What must you do to get equation (2)? Can you write another equation that would yield the same graph as $4x + 6y = 48$?

5. Summarize the results of questions 2, 3 and 4 in the following statements.
   (a) Graphs of a system that is consistent have _____ slopes. The two lines _____ .
   (b) Graphs of a system that is inconsistent have _____ slopes but _____ y-intercepts. The two lines _____ .
   (c) Graphs of a system that is dependent have _____ slopes and _____ y-intercepts. The two lines _____ . One equation is a _____ of the other equation.

6. State the conditions under which a pair of linear equations in two variables will have a unique solution.

Draw the graphs of each of the following systems of equations and tell whether each system is dependent, consistent, or inconsistent.

7. $\begin{cases} y + 2x = 3 \\ 2x - y = 1 \end{cases}$

8. $\begin{cases} 2x - 3y = 6 \\ y = 3x + 3 \end{cases}$

9. $\begin{cases} y = 3x - 2 \\ y = 3x + 4 \end{cases}$

10. $\begin{cases} 2x - y = 4 \\ 4x = 8 + 2y \end{cases}$

Solve graphically each of the following systems. Check.

11. $\begin{cases} y + 4 = 3x \\ 2y - 4x = -4 \end{cases}$

12. $\begin{cases} 3x + 2y = -1 \\ y + 2x = -7 \end{cases}$

13. $\begin{cases} -2x + y = 3 \\ y + 6 = -x \end{cases}$

14. $\begin{cases} 4x + 2y = 8 \\ y = -3x + 4 \end{cases}$

Find the solution set graphically in each of the following. Check.

15. $\{(x,y)|y = -2x\} \cap \{(x,y)|y = -4x + 6\}$

16. $\{(x,y)|y = 2x - 2\} \cap \{(x,y)|y = 4\}$

Find the solution set graphically. Do your answers check?

17. $\begin{cases} 5x + 3y = 12 \\ 2x + 6y = 11.3 \end{cases}$

18. $\begin{cases} 3x + 2y = 3 \\ -6x + 4y = 14 \end{cases}$

19. $\begin{cases} 3x + y = 70 \\ x + 2y = 75 \end{cases}$

20. $\begin{cases} y = 4x + 6 \\ y = 2x + 1 \end{cases}$

21. Do all systems of equations have solutions? To help you to answer, work through this series of questions. Graph each system. Note which have solutions and which do not.

    (a) $\begin{cases} y - x = 1 \\ 3x + 2y = 17 \end{cases}$

    (b) $\begin{cases} 2x - 3y = 12 \\ 4x - 6y = 30 \end{cases}$

    (c) $\begin{cases} x + 2y = 4 \\ 3x + 6y = 12 \end{cases}$

    (d) Study the coefficients of the variables in part (c). Then study the constants of each equation. Do you notice anything that might be pertinent?

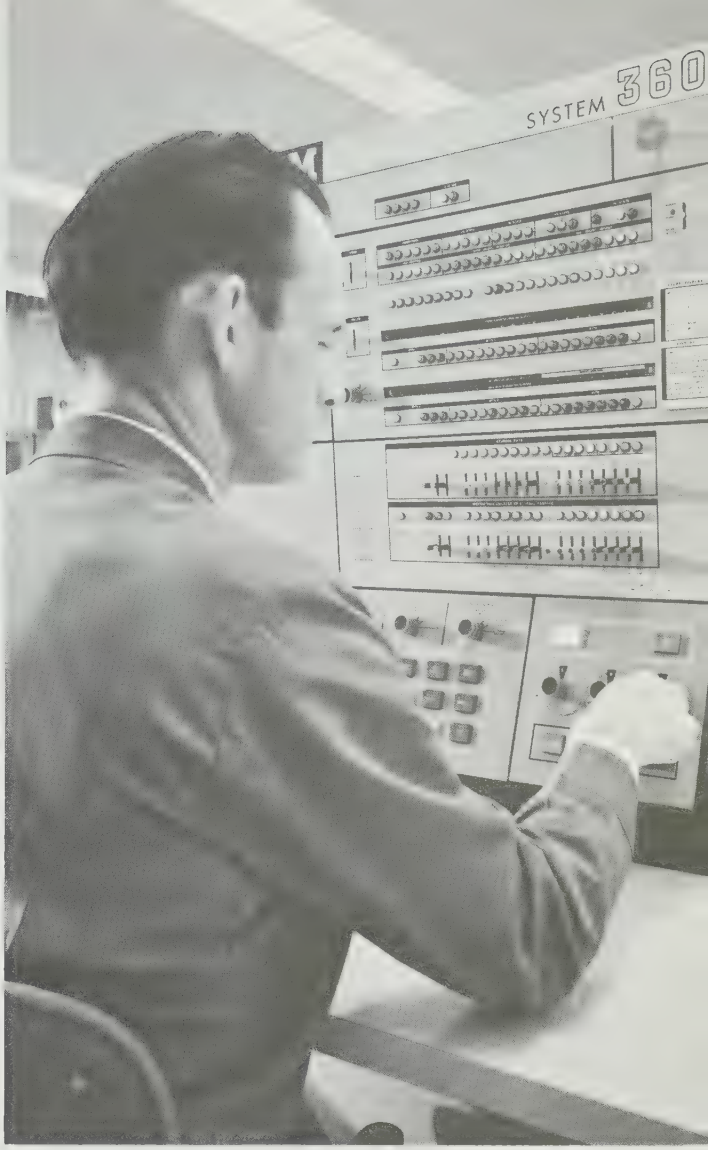
    (e) Study the coefficients of the variables in part (b). Then study the constants. Do you see a relation that was not present in part (c)?

    (f) Study the coefficients of the variables in part (a). What do you note about the coefficients and constants?

    (g) Do all systems of equations have solutions? Describe the three situations you have studied in this section. Describe the solution sets of each situation.

    (h) Make up a system of equations which will have no solution set. Give it to a student near you to see if he agrees it has no solution.

    (i) Make up a system of equations which has many solutions. Give it to a student near you to see if he agrees it has many solutions.



Computers are used to solve systems of linear equations involving many variables. A science called Linear Programming is the study of linear inequalities involving many unknown factors. Linear programming is used in many areas of the business world. The computer makes it possible to solve these problems quickly.

Electronic computers, staffed by graduates of many institutes and colleges, are used to solve a variety of problems in marketing, finance, engineering process and inventory control, as well as other areas.

6·2 Graphing Inequalities

A straight line divides the Cartesian plane into three disjoint sets of points:

the set of points in the line
the set of points above the line
the set of points below the line.

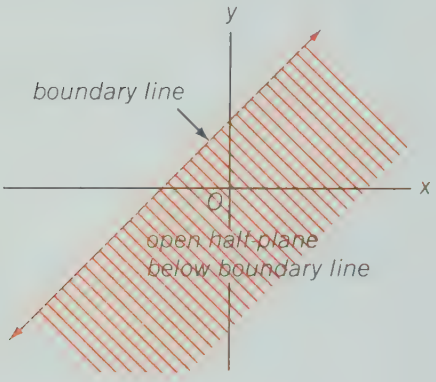


Figure 6-2

Notice we said *disjoint* sets. A half-plane does not include the points in the boundary line. We give a specific name to this half-plane that does not include the boundary line: we call it the *open half-plane.* If we wish to speak of the union of the half-plane and the boundary line, we say a *closed half-plane.* Notice how we represent an open half-plane by a broken boundary line. A closed half-plane is represented by a solid boundary line.

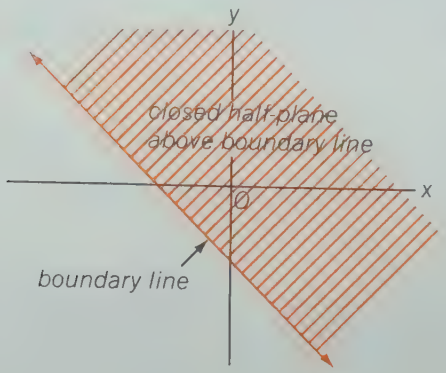


Figure 6-3

Example 1

Graph the function

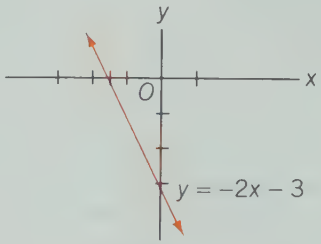
$$\{(x,y)|y \leq -2x - 3\}$$

Solution

Step 1. Write the relation as an equality.

$$y = -2x - 3$$

Step 2. Graph: $y = -2x - 3$



Step 3. Choose a point in one of the half-planes. Decide whether this point satisfies the conditions, i.e., choose (0,0). Does this point satisfy the condition $y \leq -2x - 3$?

Is $0 \leq -2 \times 0 - 3$?

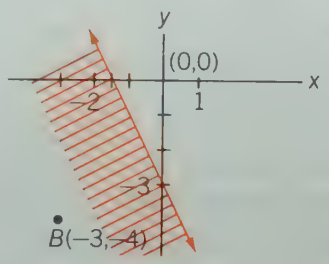
Is $0 \leq -3$? NO!

Therefore, this half-plane containing the point whose coordinates are (0,0) is not the one we want. Therefore, it must be in the other half-plane. Now choose some point in the other half-plane, i.e., $B(-3,-4)$ and test to see whether or not it satisfies the condition.

Is $-4 \leq -2(-3) - 3$?

Is $-4 \leq 3$? YES!

Therefore, B is in the required half-plane.



Step 4. The closed half-plane hashed in constitutes the solution set for $\{(x,y)|y \leq -2x - 3, y \in R\}$.

An alternate method is illustrated in the next example.

Example 2

Graph the function $\{(x,y) \mid y \leq -2x - 3\}$.

Solution

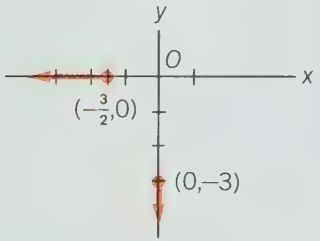
Step 1. Write the relation as an inequality where $x = 0$.

$y \leq -2x - 3$
$y \leq -3$
$\therefore$ y will equal -3 or take a value less than -3 when $x = 0$.

Step 2. Write the relation as an inequality where $y = 0$.

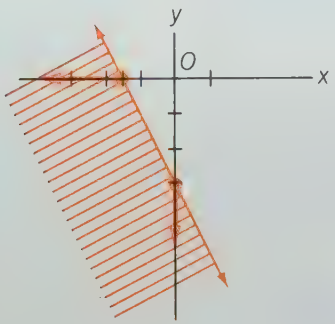
$y \leq -2x - 3$
$0 \leq -2x - 3$
$2x \leq -3$
$x \leq -\frac{3}{2}$
$\therefore$ x will equal $-\frac{3}{2}$ or take a value less than $-\frac{3}{2}$ when $y = 0$.

Step 3. Show the two inequalities on a graph.



Step 4. Join the points $(-\frac{3}{2}, 0)$ and $(0,3)$.

Step 5. Shade region in direction of arrows.

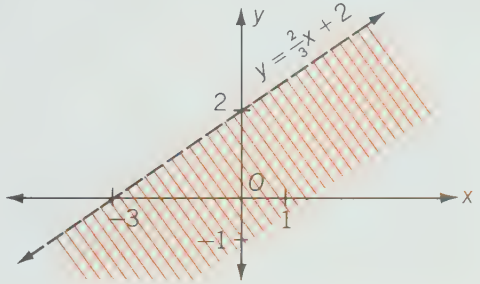


EXERCISE 6·2

1. Refer to Example 1. Draw a flow chart to show the steps in graphing a solution expressing an inequality. Choose one method.

2. What do we mean by *disjoint sets?* Name several disjoint sets.

3. (a) Is this a graph (shaded area) of an open or closed half-plane? Explain.



(b) Does (0,0) satisfy

$$y < \tfrac{2}{3}x + 2 \text{ or } y > \tfrac{2}{3}x + 2?$$

(c) Which of the following satisfy the relation shown in the graph?

(i) $y < \frac{2}{3}x + 2$ (ii) $y \leq \frac{2}{3}x + 2$
(iii) $y > \frac{2}{3}x + 2$ (iv) $y \geq \frac{2}{3}x + 2$

4. Graph the following inequalities.
   (a) $\{(x,y) \mid y < x + 2\}$
   (b) $\{(x,y) \mid y > 2x - 4\}$
   (c) $\{(x,y) \mid y \geq -3x + 5\}$
   (d) $\{(x,y) \mid y \leq 2x + 6\}$
   (e) $\{(x,y) \mid y > 6 - 2x\}$
   (f) $\{(x,y) \mid y < 3\frac{1}{2} - \frac{1}{2}x\}$
   (g) $\{(x,y) \mid y \geq \frac{1}{3}x - 5\}$
   (h) $\{(x,y) \mid y \leq -\frac{3}{4}x + 6\}$

5. Refer to question 4. Which of the graphs are unions of a half-plane and a boundary line?

6. Refer to question 4. Identify each solution set as an open or closed half-plane. Support each answer.

6·3 More Inequalities

The graph of a constant needs special attention. The relation $\{(x,y)\,|\,y = 4\}$ tells us that y has only one value; it is always 4. Since y is constant (meaning it has only one value) this is called a constant relation. On the other hand, there is no limitation on x. The variable x may take any value from the set of real numbers. See Figure 6-4.

x	−4	−3	−2	0	1	4	10
y	4	4	4	4	4	4	4

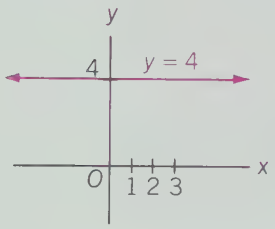


Figure 6-4

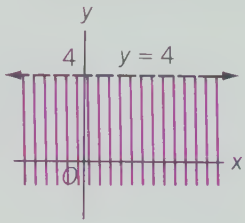


Figure 6-5

The graph of the relation $\{(x,y)|y < 4\}$ is shown in Figure 6-5. Note that we can use the point (0,0) to check.

$y < 4$ $\qquad$ $0 \,?\, 4$ $\qquad$ $0 < 4$ ✓

The point (0,0) is in the open half-plane described by $\{(x,y)|y < 4\}$.

The graph of $\{(x,y)|x \leq 4\}$ is handled in a similar manner. The line of $x = 4$ is a vertical line. Testing the point (0,0) we find

$x \leq 4$ $\qquad$ $0 \,?\, 4$ $\qquad$ $0 < 4$ ✓

Therefore, the closed half-plane we want is to the left of the line $x = 4$.

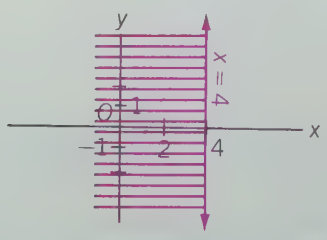
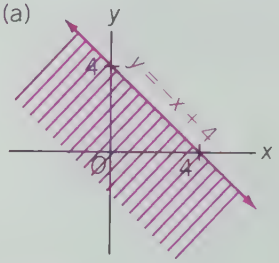


Figure 6-6

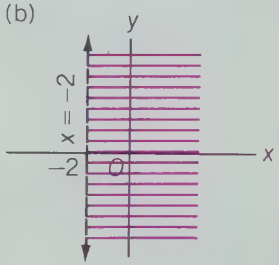
Example 1

Write the inequality represented by each figure given the equation of the boundary line.

(a)



(b)



Solution

(a) Since this is a closed half-plane (boundary line is solid) this will be either ≥ or ≤. It is the union of an inequality and an equality. Since the shaded area is below the boundary line it represents $y < -x + 4$. This can be checked by using (0,0).

$0 \,?\, -(0) + 4$
$0 < +4.$ $\qquad$ It checks.

Therefore, the relation is $\{(x,y)|y \leq -x + 4\}$.

(b) This is an open half-plane. (Why?) Therefore, it is an inequality (> or <) only. Since the area shaded is to the right of the boundary the relation is $x > -2$. (Check with (0,0)).

$x \,?\, -2$ $\qquad$ $0 \,?\, -2$ $\qquad$ $0 > -2$ ✓

The relation is $\{(x,y)|x > -2\}$.

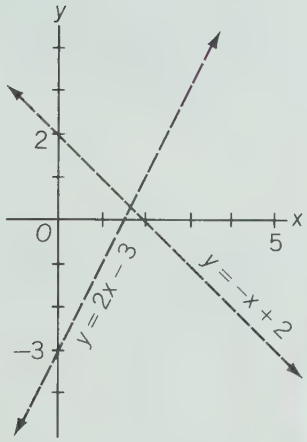
We are interested in the intersection of sets. We often want to know the solution set which satisfies two or more conditions. The next example illustrates how to graph the intersection of two sets. In the exercises you will be asked to graph the intersection of more than two sets. To do this you will show all the graphs of the conditions with respect to one set of axes.

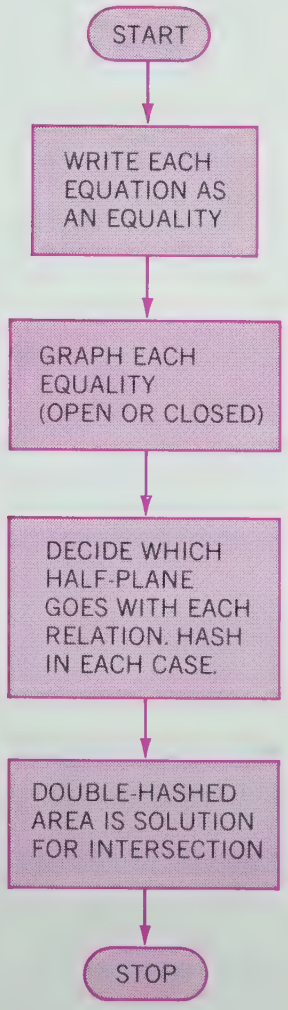
Example 2

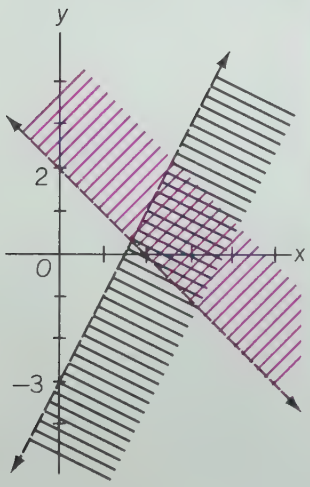
Show graphically the solution set for

$$\{(x,y)|y < 2x - 3\} \cap \{(x,y)|y > -x + 2\}$$

Solution

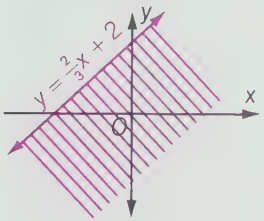


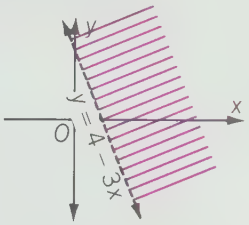

START
WRITE EACH EQUATION AS AN EQUALITY
GRAPH EACH EQUALITY (OPEN OR CLOSED)
DECIDE WHICH HALF-PLANE GOES WITH EACH RELATION. HASH IN EACH CASE.
DOUBLE-HASHED AREA IS SOLUTION FOR INTERSECTION
STOP

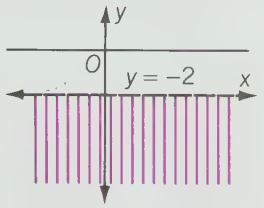


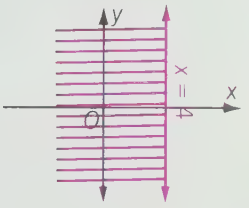
EXERCISE 6·3

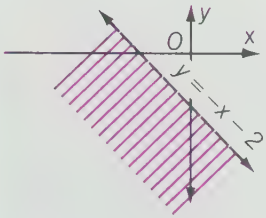
1. In each, the equation of the boundary line is given. Write the inequality represented by the half-plane (some are closed while others are open).

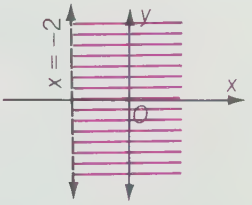












2. Graph the following relations.

   (a) $\{(x,y)|y < -3\}$ (b) $\{(x,y)|x \geq -2\}$

   (c) $\{(x,y)|x \geq 3\frac{1}{3}\}$ (d) $\{(x,y)|y \geq 2\frac{1}{2}\}$

3. Show graphically the solution set for each.

   (a) $\{(x,y)|y < 4x + 4\} \cap \{(x,y)|y > -x - 1\}$

   (b) $\{(x,y)|y \leq -2x - 3\} \cap \{(x,y)|y > \frac{1}{2}x + 2\}$

   (c) $\{(x,y)|y > 2\} \cap \{(x,y)|x \geq 2\}$

   (d) $\{(x,y)|y \geq x\} \cap \{(x,y)|y < -x\}$

   (e) $\{(x,y)|y < \frac{1}{2}x\} \cap \{(x,y)|y \geq -3x\}$

   (f) $\{(x,y)|y \geq 2\} \cap \{(x,y)|y < 2x\}$

   (g) $\{(x,y)|x < -2\} \cap \{(x,y)|x > 2\}$

   (h) $\{(x,y)|y < 3\} \cap \{(x,y)|y > -3\}$

4. Show graphically the solution set for the following.

   (a) $\{(x,y)|y > -3\} \cap \{(x,y)|y < 4\}$
   $\cap \{(x,y)|x < 4\} \cap \{(x,y)|x > -2\}$

   (b) $\{(x,y)|y < 4\} \cap \{(x,y)|y > x\} \cap \{(x,y)|y > -x\}$

5. Indicate whether each of the following is true or false.

   (a) If $u < 4$, then $u < 5$.

   (b) If $m > -3$, then $m > -4$.

   (c) If $s \leq 4$, then $s \leq -3$.

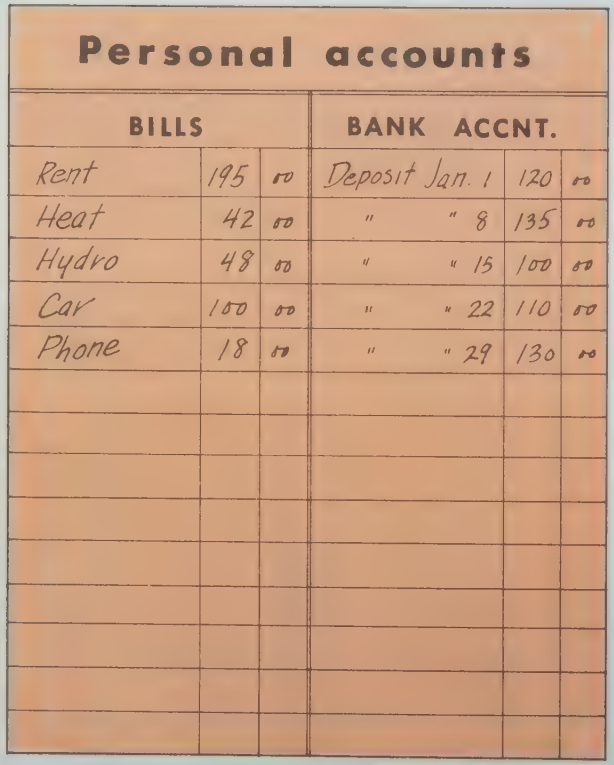
   (d) If $t \geq -9$, then $t \geq -3$.

   (e) If $n \neq -4$, then $n < 4$.

   (f) If $v \neq 6$, then $v > 6$.

   (g) If $p \neq q$, then $p < q$.

   (h) If $x < z$, then $x \geq z$.



Personal accounts

BILLS			BANK ACCNT.		
Rent	195	00	Deposit Jan. 1	120	00
Heat	42	00	" " 8	135	00
Hydro	48	00	" " 15	100	00
Car	100	00	" " 22	110	00
Phone	18	00	" " 29	130	00

Which should be the greater?

6·4 System of Linear Equations: Algebraic Approach

Questions 17 and 18 of Exercise 6·1 indicated that graphical solutions are not accurate enough when the solution involves fractions or when the numbers introduced in the questions are very large. In this chapter we extend our knowledge of algebraic solutions of systems of linear equations.

You have learned to solve a system of linear equations by the *method of comparison*. Review this method by attempting Example 1 without reading the solution. If you can do the example without assistance from the solution or from your classmates, you need not do questions 1 through 4.

Example 1

Solve the system of equations by the *method of comparison*.

$$\begin{cases} 2x + 4y = 16 \\ 2x - 8y = -11 \end{cases}$$

Solution

Have you tried this on your own?

Step 1. Solve the first equation for x in terms of y.

$$\begin{aligned} 2x + 4y &= 16 \\ 2x + 4y - 4y &= 16 - 4y && \text{(add } -4y\text{)} \\ 2x &= 16 - 4y \\ \tfrac{1}{2}(2x) &= \tfrac{1}{2}(16 - 4y) && \text{(mult. by } \tfrac{1}{2}\text{)} \\ x &= 8 - 2y \end{aligned}$$

Step 2. Solve the second equation for x in terms of y.

$$\begin{aligned} 2x - 8y &= -11 \\ 2x - 8y + 8y &= -11 + 8y && \text{(add } 8y\text{)} \\ 2x &= -11 + 8y \\ \tfrac{1}{2}(2x) &= \tfrac{1}{2}(-11 + 8y) && \text{(mult. by } \tfrac{1}{2}\text{)} \\ x &= \frac{-11 + 8y}{2} \end{aligned}$$

Step 3. Now use the transitive property of equality.

$$\text{Since } x = 8 - 2y \text{ and } x = \frac{-11 + 8y}{2}, \text{ then}$$

$$8 - 2y = \frac{-11 + 8y}{2}$$

$$8 - 2y = -\tfrac{11}{2} + 4y$$

$$8 - 2y - 4y = -\tfrac{11}{2} + 4y - 4y$$

$$8 - 6y - 8 = -\tfrac{11}{2} - 8$$

$$-6y = \tfrac{27}{2}$$

$$-\tfrac{1}{6}(-6y) = -\tfrac{1}{6}(-\tfrac{27}{2})$$

$$y = \tfrac{27}{12}$$

$$y = 2\tfrac{1}{4}$$

Step 4. In the last statement of Step 1, substitute $2\frac{1}{4}$ for y.

$$x = 8 - 2y$$

$$x = 8 - 2(2\tfrac{1}{4})$$

$$x = 3\tfrac{1}{2}$$

Step 5. The solution set: $\{(3\frac{1}{2}, 2\frac{1}{4})\}$.

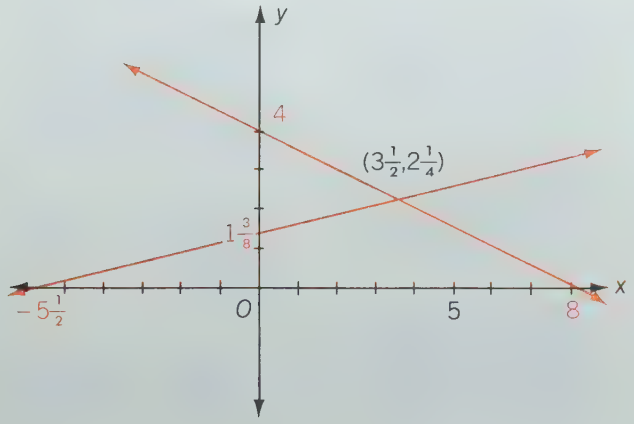
Step 6. Check, using the second equation since the first equation was used in Step 4.

$$2x - 8y = -11$$

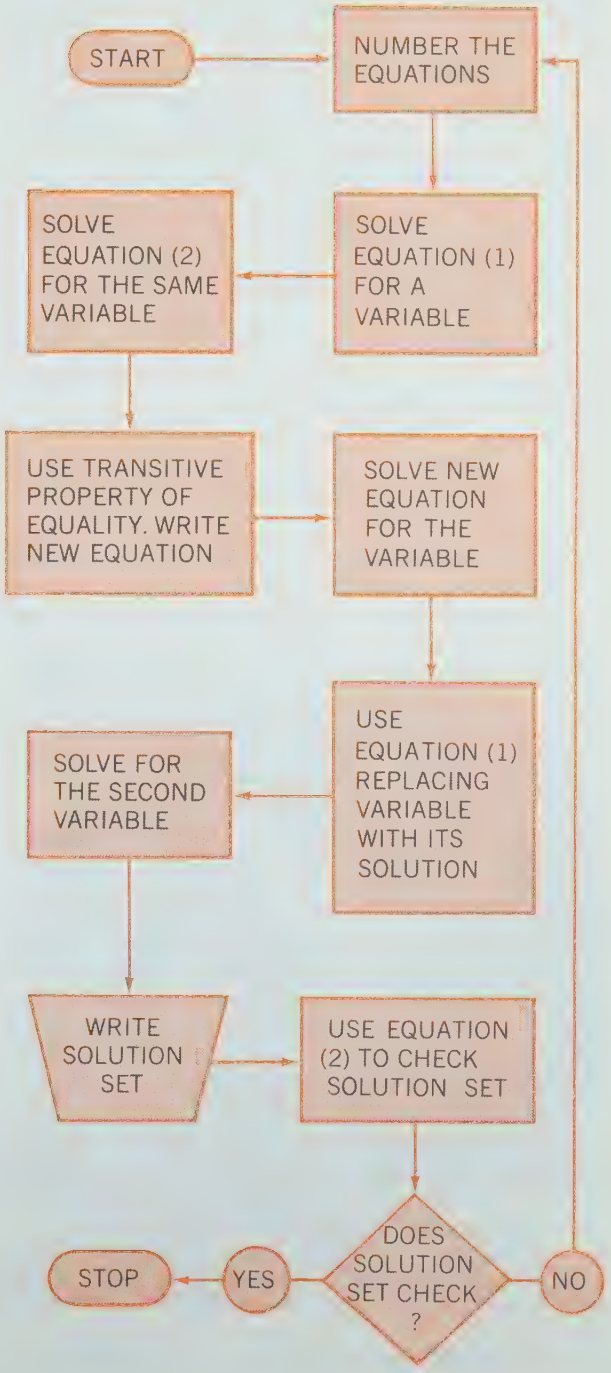
$$2(3\tfrac{1}{2}) - 8(2\tfrac{1}{4}) \stackrel{?}{=} -11$$

$$7 - 18 = -11$$

You will recall the graphical representation of this problem.



Now study the flow chart and let it help you to remember how to solve problems by this comparison method.


START
NUMBER THE EQUATIONS
SOLVE EQUATION (1) FOR A VARIABLE
SOLVE EQUATION (2) FOR THE SAME VARIABLE
USE TRANSITIVE PROPERTY OF EQUALITY. WRITE NEW EQUATION
SOLVE NEW EQUATION FOR THE VARIABLE
USE EQUATION (1) REPLACING VARIABLE WITH ITS SOLUTION
SOLVE FOR THE SECOND VARIABLE
WRITE SOLUTION SET
USE EQUATION (2) TO CHECK SOLUTION SET
DOES SOLUTION SET CHECK ?
YES
NO
STOP

EXERCISE 6·4

Find the solution set for each system of equations by the *method of comparison.* Check.

1. $x + y = 15$
   $x - y = 1$

2. $x + y = 2$
   $x + 3y = -2$

3. $x + 2y = 0$
   $x - y = 9$

4. $-2x + y = 8$
   $x - y = -3$

5. $x - y = 1$
   $3y - 2x = -4$

6. $2x - 3y = 2$
   $2y - 3x = 2$

7. $2x + 4y = 2$
   $4x - 2y = 1.5$

8. $4x - 7y = -1.6$
   $3x + 6y = 3.3$

9. $2x - 3y = -3.8$
   $3x + y = -3.5$

10. If a school club with x members levies a club fee of \$5 it receives a grant of \$$y$ from the Student Union for a total income of \$300. If it charges a \$4 fee it receives a grant of \$$(y + 50)$ to get a total of \$300. How many students are there in the club? What is the grant?

6·5 Elimination by Addition or Subtraction

To date you can solve a system of equations by

(a) graphical method

(b) comparison method.

The following examples illustrate the method of elimination. Can you see why it is called this?

Example 1

Solve the system of equations by the elimination method.

$$\{(x,y) | 2y + x = 6\tfrac{1}{2}\} \cap \{(x,y) | x - y = -1\tfrac{3}{4}\}$$

Solution

Step 1. Write the two equations one under the other so that like terms are under each other.
Change the order of the terms if necessary.

(1) $2y + x = 6\frac{1}{2}$

(2) $-y + x = -1\frac{3}{4}$

Step 2. Are the coefficients of one of the variables in the two equations numerically equal? If so, add or subtract as necessary to *eliminate* that variable. In this case the coefficients of x are equal. We will have to subtract to eliminate the x.

$$\begin{array}{rl} 2y + x = 6\frac{1}{2} & \\ -y + x = -1\frac{3}{4} & \textit{Subtract} \\ \hline 3y \quad = 8\frac{1}{4} & \end{array}$$

Step 3. Solve for the other variable (y).

$$y = 8\tfrac{1}{4} \times \tfrac{1}{3}$$
$$y = \tfrac{11}{4}$$

Step 4. Use this value to replace y in equation (2). Solve for the second variable x.

$$-\tfrac{11}{4} + x = -1\tfrac{3}{4}$$
$$x = \tfrac{11}{4} - \tfrac{7}{4}$$
$$x = 1$$

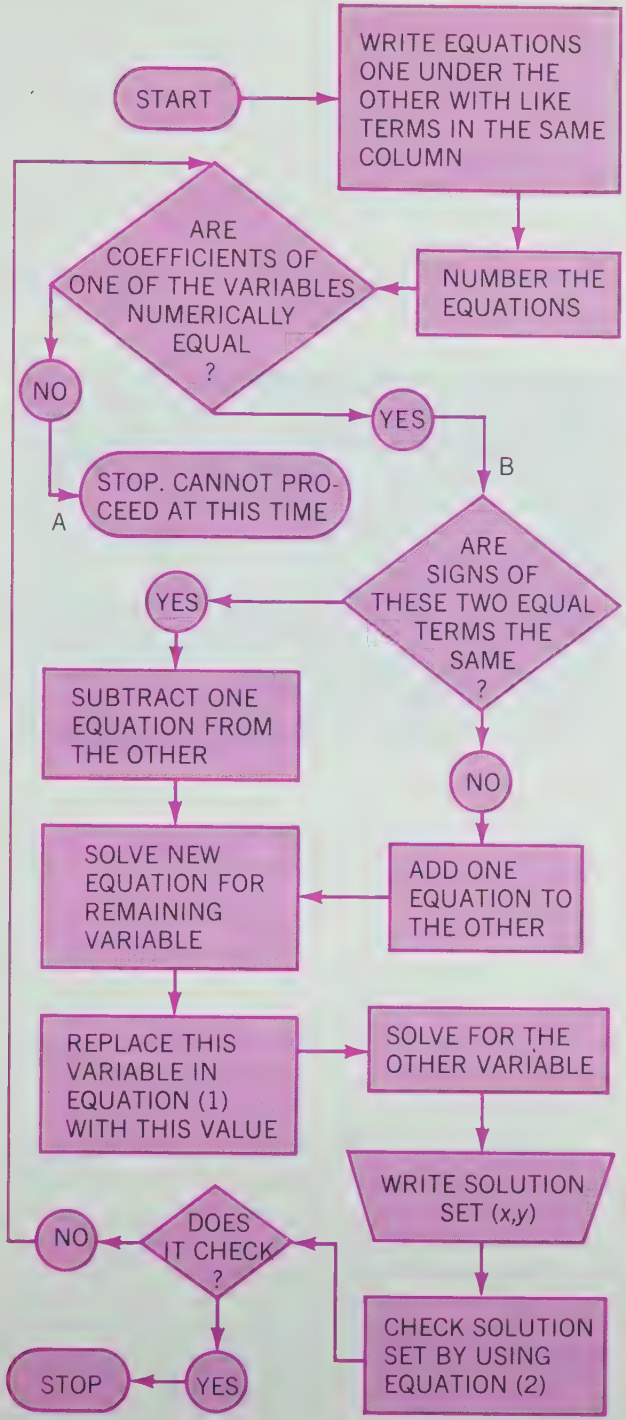
Step 5. Record the solution set.

$$\{(1, \tfrac{11}{4})\}$$

Step 6. Check by substituting the solution set in equation (1).

$$2y + x = 6\tfrac{1}{2}$$
$$2 \cdot \tfrac{11}{4} + 1 \stackrel{?}{=} 6\tfrac{1}{2}$$
$$\tfrac{11}{2} + 1 \stackrel{?}{=} 6\tfrac{1}{2}$$
$$\tfrac{13}{2} = 6\tfrac{1}{2}$$

Flow Charting For The Elimination Method


START
WRITE EQUATIONS ONE UNDER THE OTHER WITH LIKE TERMS IN THE SAME COLUMN
NUMBER THE EQUATIONS
ARE COEFFICIENTS OF ONE OF THE VARIABLES NUMERICALLY EQUAL ?
NO
YES
A
STOP. CANNOT PROCEED AT THIS TIME
B
ARE SIGNS OF THESE TWO EQUAL TERMS THE SAME ?
YES
NO
SUBTRACT ONE EQUATION FROM THE OTHER
ADD ONE EQUATION TO THE OTHER
SOLVE NEW EQUATION FOR REMAINING VARIABLE
REPLACE THIS VARIABLE IN EQUATION (1) WITH THIS VALUE
SOLVE FOR THE OTHER VARIABLE
WRITE SOLUTION SET (x,y)
DOES IT CHECK ?
NO
YES
STOP
CHECK SOLUTION SET BY USING EQUATION (2)

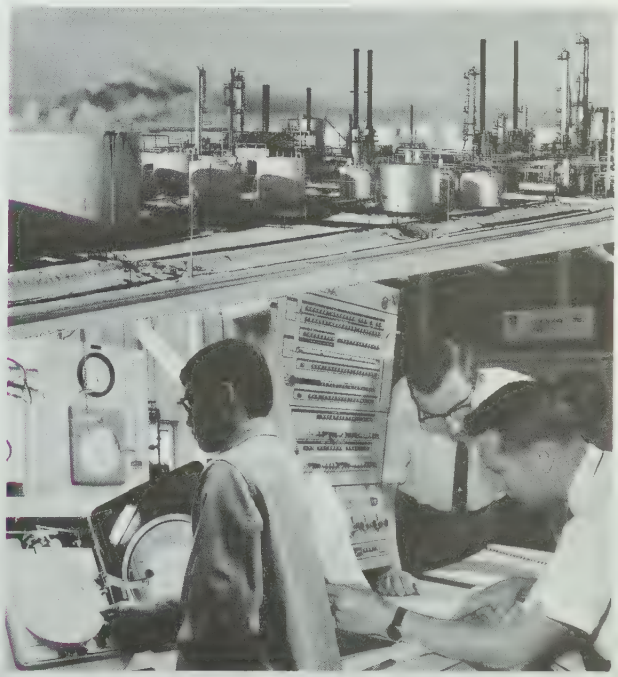
EXERCISE 6·5

Use the Elimination Method to find the solution set. Check.

1. $\{(x,y)|3x + y = 1\} \cap \{(x,y)|2x - y = 4\}$
2. $\{(x,y)|x - 2y = 6\} \cap \{(x,y)|x - 7y = -9\}$
3. $\{(x,y)|2x - 2y = 6\} \cap \{(x,y)|3x + 2y = \frac{28}{3}\}$
4. $\{(s,t)|2s + t = 23\} \cap \{(s,t)|2s + 4t = -4\}$
5. $\{(m,n)|m - n = 2\} \cap \{(m,n)|m + n = 100\}$

Solve each system. Check.

6. $x + 2y = 5$
   $x - 3y = 40$
7. $x + y = 2$
   $3x - y = \frac{1}{2}$
8. $x - 2y = -1.1$
   $x + 3y = 2.4$
9. $x + y = 6$
   $x - y = -\frac{6}{5}$
10. $x - y = -11$
    $x + y = 27$
11. $x + 4y = 1$
    $x + 6y = 5$



Industry uses systems of equations to calculate the most profitable way to conduct its business.

6·6 Elimination by Addition or Subtraction Continued

Did you notice in the previous exercises that you needed a variable that had the same coefficient in each equation? Not every system of equations has this condition. Sometimes it is necessary to write an equation equivalent to a given equation in order to obtain this situation. (Reread pages 107-108.) Observe this in the next example.

Example 1

Solve:

$$\{(x,y)|3x + 2y = 1\tfrac{1}{2}\} \cap \{(x,y)|x + 3y = -3\}$$

Solution

Step 1. Write the two equations one under the other.

(1) $3x + 2y = 1\frac{1}{2}$

(2) $x + 3y = -3$

Step 2. Let us eliminate the x variable. Write an equivalent equation for (2) such that the coefficient of x is 3, the same as it is in (1).

(2) $x + 3y = -3$

Multiply both portions of equation by 3.

(3) $3x + 9y = -9$

Step 3. Now write equations (1) and (3) again.

(1) $3x + 2y = 1\frac{1}{2}$

(3) $3x + 9y = -9$

Step 4. Subtract to eliminate the x's

$$-7y = 10\tfrac{1}{2}$$

Step 5. Solve for y.

$$y = \tfrac{21}{2}(-\tfrac{1}{7})$$
$$y = -\tfrac{3}{2}$$

Step 6. Use equation (2). Replace y with $-\frac{3}{2}$.

$$x + 3y = -3$$
$$x + 3(-\tfrac{3}{2}) = -3$$

Step 7. Solve for x

$$x - \tfrac{9}{2} = -3$$
$$x = \tfrac{3}{2}$$

Step 8. Write the solution set.

$$\{(\tfrac{3}{2}, -\tfrac{3}{2})\}$$

Step 9. Check by using equation (1).

$$3x + 2y = 1\tfrac{1}{2}$$
$$3(\tfrac{3}{2}) + 2(-\tfrac{3}{2}) \stackrel{?}{=} 1\tfrac{1}{2}$$
$$\tfrac{9}{2} - \tfrac{6}{2} \stackrel{?}{=} 1\tfrac{1}{2}$$
$$\tfrac{3}{2} = 1\tfrac{1}{2}$$

Example 2

Write an equation equivalent to the one given. The new equation is to contain the terms given.

(a) $x + 2y = 9$; $3x$

(b) $3y = 9 + x$; $-5x$

Solution

(a) The new equation must have the term $3x$. We will have to multiply each part of the equation by 3. The new equation is equivalent to $x + 2y = 9$.

Do you recall graphing dependent systems in Section 6·1? Refer again to this section. Recall that all points (solution sets) that satisfied one equation satisfied the second equation. The two equations of a dependent system are equivalent.

$$x + 2y = 9$$
$$3x$$
$$3(x + 2y) = 3 \times 9$$
$$3x + 6y = 27$$

(b) We wish to multiply equation (1) by -5 so that the coefficient of x is -5. Equation (2) is equivalent to equation (1).

$$\begin{array}{ll} (1) & 3y = 9 + x \\ & -5x \\ & -5(3y = 9 + x) \\ (2) & -15y = -45 - 5x \end{array}$$

EXERCISE 6·6

In each question an equation is given. Write an equivalent equation using the one term given.

1. $x + 3y = 9$; $3x$
2. $2x - y = -6$; $5y$
3. $6x + y = 9$; $-4y$
4. $y = 5x + 4$; $13y$
5. $4x = y - 7$; $6y$
6. $6x - 2y = 8$; y

Use the Elimination Method to solve each. Check.

7. $2x + 3y = -9$; $x - 2y = -1$
8. $3v + 2t = -9$; $v - t = -28$
9. $a - 2b = 26$; $4a + 5b = 13$
10. $4a + 3b = 20$; $b - 3a = \frac{-19}{3}$
11. $2y + 5x = 11$; $x + 3y = -3$
12. $2y + 5x = 9$; $x - 3y = -5$
13. $\frac{x}{2} + \frac{y}{3} = 6$; $x + \frac{y}{3} = 11$
14. $3y = 4x$; $y - 2x = -4$

6·7 Elimination by Addition or Subtraction Concluded

In some cases we will have to write an equivalent equation for each of the equations given. Follow the explanation in Example 1.

Example 1

Solve the system:

$$\begin{array}{l} 2x + 5y = 7 \\ 3x + 4y = 7 \end{array}$$

Solution

In order to eliminate the x terms we will have to write each equation so that the coefficients of the x terms are both 6, since 6 is the *lowest common multiple* of 2 and 3.

$$\begin{array}{ll} (1) & 2x + 5y = 7 \\ (2) & 3x + 4y = 7 \end{array}$$

In order to write the first equation with an x-coefficient of 6 we shall have to multiply the whole equation by 3.

$$\begin{array}{ll} & 3(2x + 5y = 7) \\ (3) & 6x + 15y = 21 \end{array}$$

In order to write equation (2) with an x-coefficient of 6 we shall have to multiply the whole equation by 2.

$$\begin{array}{ll} (2) & 3x + 4y = 7 \\ & 2(3x + 4y = 7) \\ (4) & 6x + 8y = 14 \end{array}$$

Now we shall use equations (3) and (4) to eliminate the x terms. We shall subtract.

$$\begin{array}{ll} (3) & 6x + 15y = 21 \\ (4) & 6x + \ 8y = 14 \end{array}$$

Can you complete this example? The answer is (1,1).

EXERCISE 6·7

In each question an equation is given. Write an equivalent equation using the one term given.

1. $3x + 2y = 7$; $6x$
2. $2y - 3x = 4$; $6y$
3. $4x - 3y = 5$; $12x$
4. $7x - 13y = -3$; $21x$
5. $13x - 5y = -4$; $30y$
6. $6y - 11x = 4$; $77x$

7. State the lowest common multiple for each pair of numbers.

(a) (4,5) (b) (2,6) (c) (3,6)
(d) (4,6) (e) (6,8) (f) (11,6)
(g) (3,7) (h) (15,3) (i) (8,12)

Use the Elimination Method to find the solution set of each system.

8. $3a + 2b = 13$
$8a + 3b = 9$

9. $4m + 3n = -11$
$7m - 4n = 64$

10. $7x - 6y = 5$
$5y - 6x = -6$

11. $4a + 6t = -23$
$3a + 5t = -47$

12. $2y + 5x = 11$
$4y + 3x = 1$

13. $2y + 5x = 9$
$3y + 2x = 8$

Find the solution set by the Method of Elimination.

14. $\{(x,y)|5x - 10y = -25\} \cap \{(x,y)|6y - 4x = \frac{18}{5}\}$

15. $\{(x,y)|3x + 2y = 2\} \cap \{(x,y)|6x - 9y = -16\}$

6·8 Problems Involving Systems of Linear Equations

We shall now solve problems by using two variables. In each question we shall identify two unknowns and let them be represented by any two symbols. Then we shall write two equations using the variables. Next we shall solve the system of equations. Finally, we shall check our answers.

Example 1

The sum of 2 numbers is 58. Three times the smaller is 2 less than the larger number. Find the two numbers.

Solution

Step 1. Identify the two unknowns.

Let s represent the smaller number.
Let t represent the larger number.

Step 2. Write the two equations using the variables.

The sum of two numbers is 58.

(1) $s + t = 58$

3 times the smaller number is 2 less than the larger number.

$3s = t - 2$

(2) $3s + 2 = t$

Step 3. Solve the system of equations.
We will use the replacement method.

(1) $s + t = 58$
$s = 58 - t$

(2) $3s + 2 = t$
$3(58 - t) + 2 = t$
$174 - 3t + 2 = t$
$4t = 176$
$t = 44$

(1) $s + t = 58$
$s + 44 = 58$
$s = 14$

$\{14,44\}$

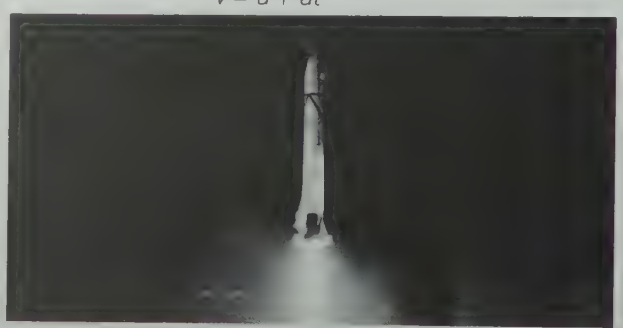
Step 4. Answer the question. The two numbers are 14 and 44. Can you check this?

Example 2

A rocket has an initial velocity of 6 miles per minute. It accelerates at the rate of 10 miles per minute per minute. Another rocket has an initial velocity of 20 miles per minute with an acceleration of 7 miles per minute per minute. If the two rockets fire simultaneously, how many minutes will it be before the two rockets have the same velocity? What will that velocity be?

Hint! Final velocity = initial velocity + acceleration × time

$v = u + at$



Solution

Step 1. Identify two unknowns.

Use the variables v and t.

Let v represent the final velocity in miles per minute.

Let t represent the time in minutes.

Step 2. Write the two equations using the variables.

(1) $v = 6 + 10t$ (Initial velocity is 6 and acceleration is 10.)

(2) $v = 20 + 7t$ (Initial velocity is 20 and acceleration is 7.)

Step 3. This question is set up for the *Comparison Method.* Do you see why?

$$6 + 10t = 20 + 7t$$
$$3t = 14$$
$$t = \tfrac{14}{3}$$
$$v = 6 + 10 \times \tfrac{14}{3}$$
$$v = \tfrac{158}{3}$$
$$v = 52\tfrac{2}{3}$$

Step 4. Answer the questions.

The two rockets will have the same velocity after $\frac{14}{3}$ minutes. The velocity at that time will be $52\frac{2}{3}$ miles per minute. Check this.

Steps in Problem Solving

1. Assign two variables to the two unknowns.
2. Set up two equations.
3. Solve the system of equations.
4. Interpret the solution set to answer the questions.

EXERCISE 6·8

Write a pair of equations in two variables for the facts given in the situations described.

1. The perimeter of a rectangle is 88. The length is 8 units longer than twice its width.
2. The measure of one of the equal sides of an isosceles triangle is twice the measure of the base. The perimeter of the triangle is 55.
3. The first number is 2 more than twice the second number. The sum of twice the larger and three times the smaller is 221.
4. A man invests two sums of money totalling $900. He invests some at 5% and the balance at 10%. His annual interest on the investment is $72.50.
5. Two angles are complementary. The measure of the larger angle is 4 more than 3 times the measure of the smaller angle.
6. The sum of Mary's age and her mother's is 58. In 10 years her mother will be twice as old as Mary.
7. In an election the 3450 votes were split between the new candidate and the incumbent. If the incumbent had received five times as many votes, he would have received 425 less than twice as many as the new candidate received.

8. Solve each system of equations in questions 1 through 7.

Use two variables and two equations to solve each of the problems.

9. Jack and Tom together have $200. If Jack had twice as much and Tom had three times as much, they would together have enough to buy a skidoo for $520. How much does each have now?

10. Two numbers differ by 36. The larger exceeds twice the smaller by 4. Find the two numbers.

11. Wall-Street Jack invested $300, part at 4% and the balance at 6%. He earned a total of $15.40 from his investment in interest in one year. How much did he invest at each interest rate?

12. A plane leaves the runway with a velocity of 2 mi/min. Its acceleration is constant at 4.5 mi/min/min. A second plane, leaving the runway at the same instant, has a velocity of 5 mi/min. Its acceleration is 2.25 mi/min/min. After what time will the two planes have the same velocity? What is this velocity?

Using the Library

1. *The Giant Golden Book of Mathematics.* Read pages 63-65. Draw a graph to show the distance a falling body falls in the first 7 seconds of its fall.

2. *Romping Through Mathematics.* Read pages 117-120. What do we mean by "flying off on a tangent"?

REVIEW EXERCISE 6

1. Show graphically the solution set for

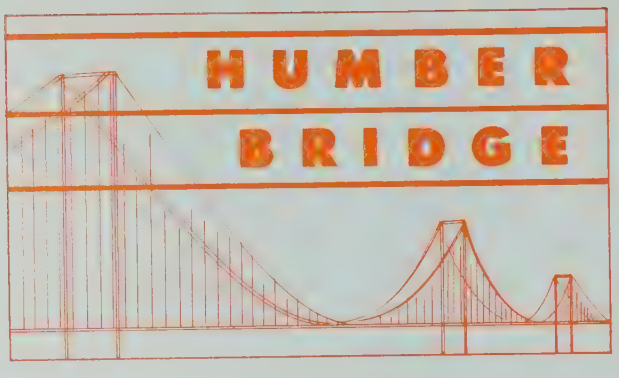
$$3x + y = 5$$
$$x - y = 1$$

Check the solution by solving the system algebraically.

2. In shopping for sports equipment, Ellen noted the cost of three tennis balls was 25 cents more than the cost of three golf balls. She also noted that the cost of four golf balls was 25 cents more than the cost of three tennis balls. What was the price of a single tennis ball and a single golf ball? Use two variables and two equations to solve.

3. Solve algebraically:

$$2x + 5y = -11$$
$$3x + 4y = -6.$$



The Humber Bridge Company fabricates a number of products such as buildings, bridges, tanks, vessels and hoppers, all of which require welding. One of the latest welding processes is the automatic submerged arc process, in which the welding machine travels along the material automatically while welding wire from a reel is melted under a cover of flux flowing over the weld.

The cost of welding is comprised of the cost of the welding material and the time taken to perform the welding.

The deposition rate of weld material varies directly with the amperage. The deposition rate for our machines and the wire used is 0.50 lb per minute at 1000 amps.

The allowable amperage is dependent on the material thickness, material preparation, configuration and gap or space between pieces so that the arc will not blow through.

Problem

It is required to weld two 6-ft diameter cylinders together, to form one cylinder. The cross section area of the weld is 0.25 square inches and the weld must be made in two passes, a root pass and a filler pass.

The root pass can be placed at 500 amps and at 12 inches per minute. The filler pass can be placed at 800 amps.

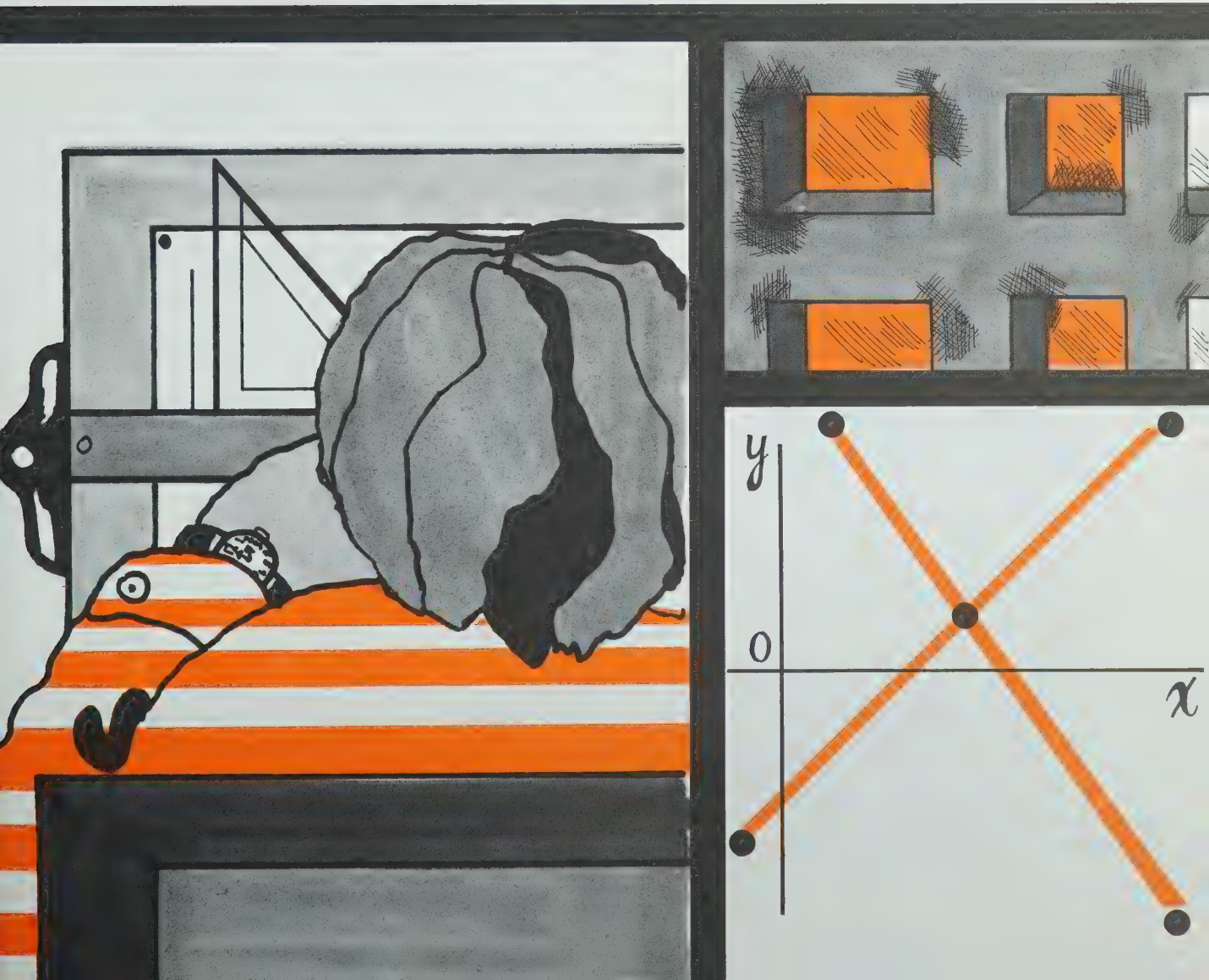
Metal weight is 0.283 lb per cubic inch.

Find:

(a) Weight of metal deposited;

(b) How long it will take.

Chapter Seven
ANALYTICAL GEOMETRY



7·1 Introduction

Until the seventeenth century algebra and geometry were two distinct areas of study. René Descartes (1596-1650) introduced a new method of solving geometrical problems. This method linked algebra and geometry by means of the coordinate system which associates points with numbers and numbers with points. This field of study—called analytical geometry—allows us to study geometric relations by means of algebraic statements. In this chapter we find algebraic expressions for

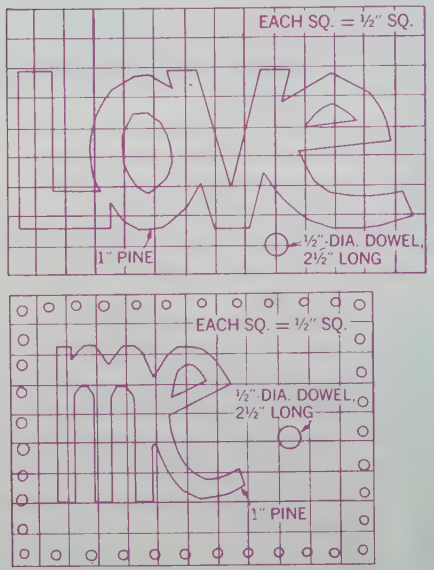
(1) the distance between two points (or the measure of the segment whose endpoints are known);

(2) the coordinates of the midpoint of a segment;

(3) determining whether three points (in the Cartesian plane) are collinear.

POP-ART PEGS

Cut blocks from 1 x 5 pine. Follow patterns for letters and cut from pine scrap, using a saber or coping saw. Cut dowels and round ends. Drill holes at a slight angle. Sand blocks and lettering; smooth edges. Outline letters on blocks; paint all parts except inside outlines. Glue on letters and dowels with white glue. Attach 2 metal picture hangers to each block.



The Cartesian plane is often used to enlarge diagrams or plans.

7·2 Distance Between Two Points

For every point in the Cartesian plane there is a unique ordered pair of numbers. For example, in Figure 7-1 the point A is determined by the ordered pair (3,2). Conversely, for each ordered pair of numbers there is one and only one point. A segment is determined by a pair of distinct points, hence by a pair of distinct ordered pairs. In Figure 7-1 the two ordered pairs (−4,1) and (2,−3) determine the segment $\overline{PQ}$. We shall indicate the measure of the length of the segment $\overline{PQ}$ by $m(\overline{PQ})$.

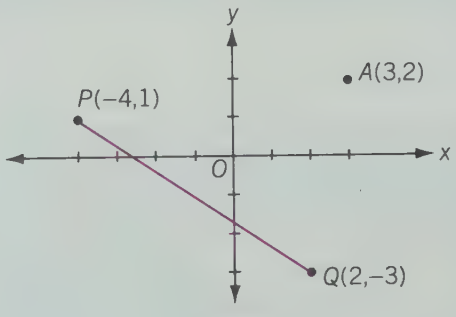
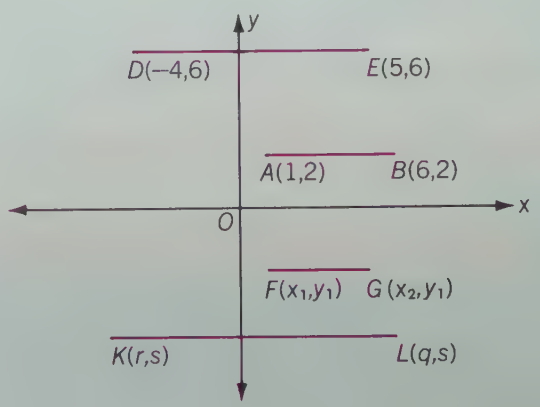


Figure 7-1

EXERCISE 7·2

1. Study the diagram below. Record all the information you know or can find about each of the segments: $\overline{AB}, \overline{DE}, \overline{FG}, \overline{KL}$.



Does the information you recorded in question 1 contain the answers to questions 2 and 3? If not, answer questions 2 and 3. Refer to the diagram for question 1.

2. For each of the segments $\overline{AB}, \overline{DE}, \overline{FG}, \overline{KL}$, what is noticeable about the y-coordinates of the endpoints as opposed to the x-coordinates?

3. Find $m(\overline{AB})$, $m(\overline{DE})$, $m(\overline{FG})$, $m(\overline{KL})$. How did you find the measure in each case? Did you count? If you did, try to find another method.
Hint! Subtract.

4. Find $m(\overline{MN})$ given $M(5,2)$ and $N(-2,2)$.

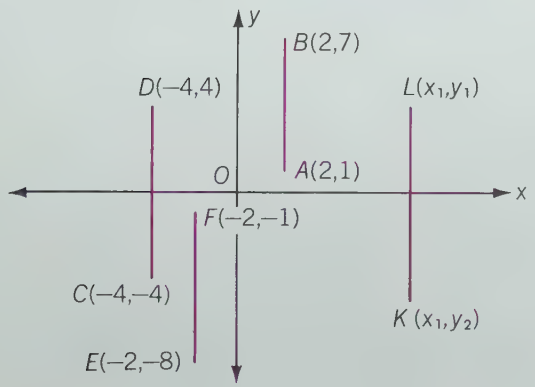
5. What is the measure of the segment whose endpoints are $(-2,-3)$ and $(-5,-3)$?

6. If you are given any two points (x_1,y_1) and (x_2,y_2) in the Cartesian plane, do you know which is the larger number, x_1 or x_2? Can you tell whether $x_2 - x_1$ is positive or negative? How can we write $x_2 - x_1$ to ensure it is always positive?

7. You may recall that the measure of a segment is always a positive number. How shall we write $x_2 - x_1$ to ensure that it is always a positive number? Review Section 5-15.

8. The measure of the segment whose endpoints are (x_1,y_1) and (x_2,y_1) is _____. This segment is _____ to the x-axis.

9. Study the figure below. Find the measure of each segment. Have you indicated the measure of $\overline{KL}$ as a positive number?

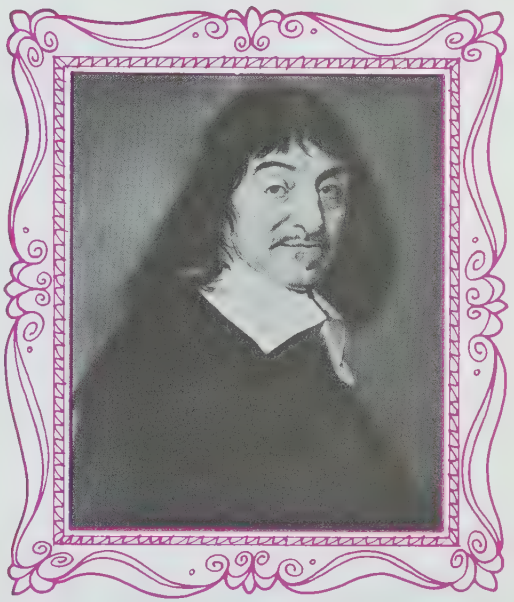


10. What is the measure of each segment determined by each pair of endpoints.
(a) (3,1), (3,5)
(b) (−2,−2), (−2,−4)
(c) (4,−3), (4,6)
(d) (x_2,y_2), (x_2,y_1)
(e) (m,n), (m,p)
(f) (g,h), (g,k)

11. The measure of the segment whose endpoints are (x_2,y_2) and (x_2,y_1) is _____. This segment is _____ to the y-axis.

12. Look up in the dictionary the word *analytical*. How does the meaning of analytical apply to the study of mathematics as we have it in this chapter?

13. Find the derivation of the name *Cartesian plane*.



7·3 Distance Between Any Two Points

From the questions in Exercise 7·1 you will have concluded the measure of a segment parallel to the x-axis is given by $|x_2 - x_1|$. The measure of a segment parallel to the y-axis is $|y_2 - y_1|$. The absolute value of the difference between the x-coordinates or the y-coordinates assures the measure is positive. Hence, there is no difference between $|x_2 - x_1|$ and $|x_1 - x_2|$ nor between $|y_2 - y_1|$ and $|y_1 - y_2|$.

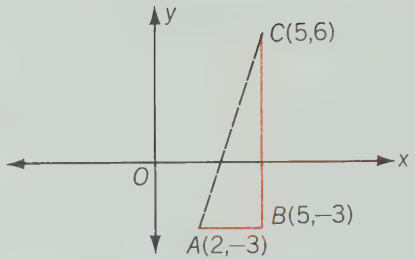


Figure 7-2

In Figure 7-2 $m(\overline{AB}) = |2 - 5|$
$= |-3|$
$= 3.$

Similarly, $m(\overline{BC}) = |6 - (-3)|$
$= |6 + 3|$
$= 9.$

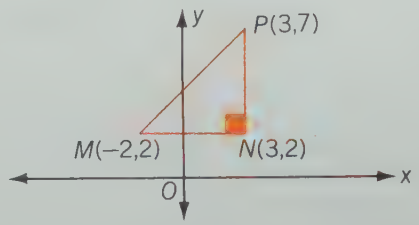
To find $m(\overline{AC})$ we now use the Pythagorean Property:

$$m(\overline{AC}) = \sqrt{m(\overline{AB})^2 + m(\overline{BC})^2}$$
$$= \sqrt{3^2 + 9^2}$$
$$= \sqrt{90}$$
$$= \sqrt{9}\,\sqrt{10}$$
$$= 3\sqrt{10}.$$

$$d = |y_2 - y_1|$$

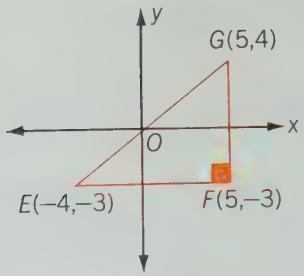
EXERCISE 7·3

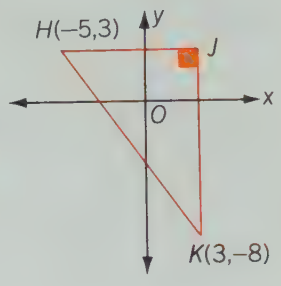
1. Refer to the diagram. Find $m(\overline{AC})$ given $m(\overline{AB}) = 4$, $m(\overline{BC}) = 7$.



2. Refer to the diagram above. Find $m(\overline{MN})$, $m(\overline{NP})$ and $m(\overline{MP})$. Plot the points on a graph and check by measuring.

3. Refer to $\triangle EFG$. Find the measure of each side.



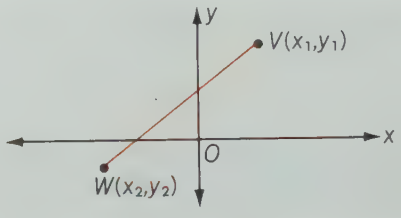


4. Refer to $\triangle HJK$. What are the coordinates of point J? Find $m(\overline{HJ})$, $m(\overline{JK})$ and $m(\overline{HK})$.

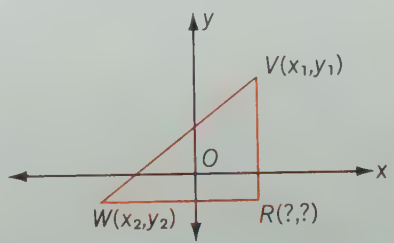
5. The endpoints of a segment are (1,–3) and (7,8). Find the measure of the segment. (If necessary, draw the segment on a Cartesian plane and draw in the two sides to form a right triangle.)

6. Find the measure of the segment determined by the pair of endpoints whose coordinates are
   (a) (–5,2), (7,6) (b) (–6,6), (4,–3)
   (c) (6,4), (–2,–1) (d) (–11,2), (4,–6).

7. Given $V(x_1,y_1)$ and $W(x_2,y_2)$ find $m(\overline{WV})$. If you need help work through the questions that follow.



   (a) In the diagram below we have drawn in the two sides to form a right triangle. What are the coordinates of R?



   (b) What is $m(\overline{WR})$? (c) What is $m(\overline{VR})$?
   (d) What is $m(\overline{WV})$?

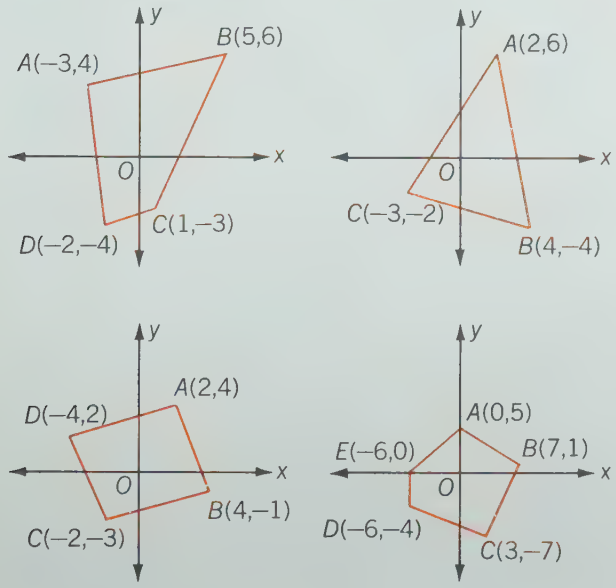
8. You learned from question 7 the measure d of a segment whose endpoints are (x_1,y_1) and (x_2,y_2) is
$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$
Find the measure of each segment given the endpoints
   (a) $P(3,5)$, $Q(2,7)$; $\overline{PQ}$
   (b) $R(0,0)$, $S(5,-12)$; $\overline{RS}$
   (c) $T(-9,6)$, $V(-3,0)$; $\overline{TV}$
   (d) $V(-3,0)$, $T(-9,6)$; $\overline{VT}$
   (e) $B(7,6)$, $D(-3,-3)$; $\overline{BD}$

9. Repeat 8(a), 8(b) using the formula
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$
Are the answers the same as in 8(a), 8(b)? Why is the quantity $|x_1 - x_2|^2$ equal to $(x_1 - x_2)^2$?
Hint! Why is $|-3|^2$ the same as $(-3)^2$?

10. Write a flow chart for finding the measure of a segment on the Cartesian plane given the coordinates of the endpoints.

11. Find the perimeter of a triangle whose vertices are $A(3,0)$, $B(4,-5)$ and $C(6,7)$.

12. Find the perimeter of the following figures, correct to 3 decimal places. Use tables or slide rule for evaluating square roots.



7·4 Midpoint of a Segment

You now know the length of any segment whose endpoints are $M(x_1,y_1)$ and $N(x_2,y_2)$ is
$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

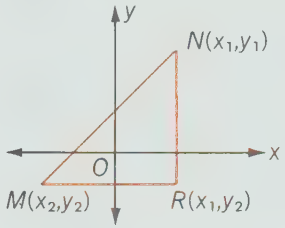


Figure 7-3

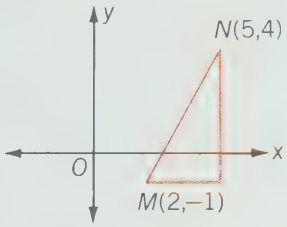


Figure 7-4

If the two endpoints are $M(2,-1)$ and $N(5,4)$ then, by the Pythagorean property, the measure of $\overline{MN}$ is

$$\begin{aligned} m(\overline{MN}) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2-5)^2 + (-1-4)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34}. \end{aligned}$$

We now turn our attention to the midpoint of a segment in the Cartesian plane given the coordinates of endpoints. We note, in Figure 7-5, that $m(\overline{PQ})$ is 8 units.

$$\begin{aligned} m(\overline{PQ}) &= |x_1 - x_2| \\ &= |-2 - 6| \\ &= |-8| \\ &= 8 \end{aligned}$$

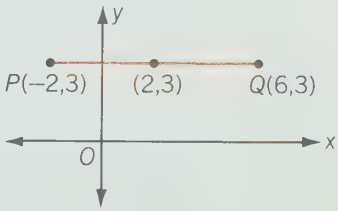


Figure 7-5

Now, by counting 4 units (half of 8) to the right from point P, we find the coordinates of the midpoint are $(2,3)$. The x-coordinate is 2. This can be found by:

$$\frac{-2+6}{2} = 2.$$

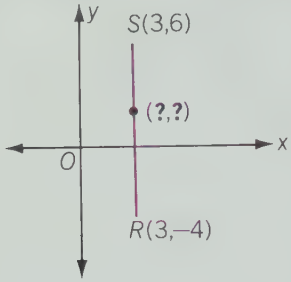
From this one case we conclude that the x-coordinate of the midpoint of a line is one-half the sum of the x-coordinates of the endpoints: $\frac{x_1 + x_2}{2}$.

Similarly, if the line is parallel to the y-axis the y-coordinate of the midpoint of the segment is $\frac{y_1 + y_2}{2}$.

Example 1

Given: $R(3,-4)$ and $S(3,6)$.

What is the x-coordinate of the midpoint of $\overline{RS}$?
What is the y-coordinate of the midpoint of $\overline{RS}$?
What are the coordinates of the midpoint of $\overline{RS}$?



Solution

All points in the segment have the same x-coordinate: 3. The y-coordinate is given by

$$\frac{y_1 + y_2}{2} = \frac{-4 + 6}{2}$$
$$= 1$$

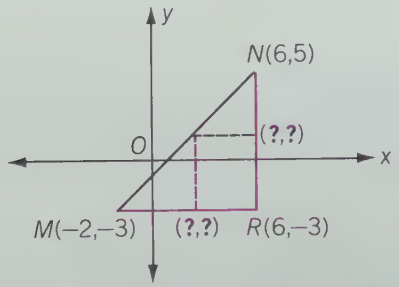
The coordinates of the midpoint are (3,1).

Example 2

Find the midpoint of $\overline{MN}$ given $N(6,5)$ and $M(-2,-3)$.

Solution

Draw the diagram completing the right triangle as shown.



The midpoint of $\overline{MN}$ will have the same x-coordinate as the midpoint of $\overline{MR}$. How do we know this?

Hence
$$\frac{x_1 + x_2}{2} = \frac{-2 + 6}{2}$$
$$= 2.$$

The midpoint of $\overline{MN}$ will have the same y-coordinate as the midpoint of $\overline{RN}$. Why?

Hence
$$\frac{y_1 + y_2}{2} = \frac{-3 + 5}{2}$$
$$= 1$$

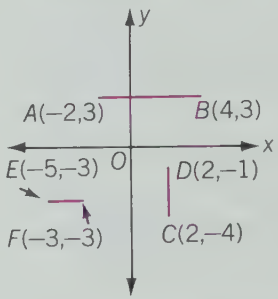
The midpoint of $\overline{MN}$ is (2,1).

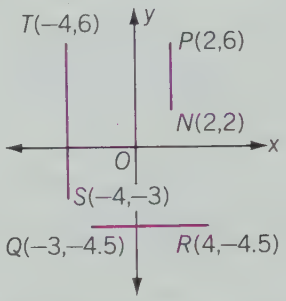
From the above example we conclude that the midpoint of a segment whose endpoints are (x_1,y_1) and (x_2,y_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

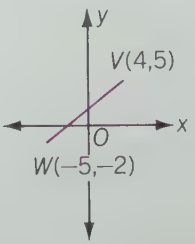
EXERCISE 7·4

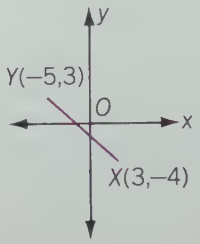
1. Give orally the midpoint of each segment.

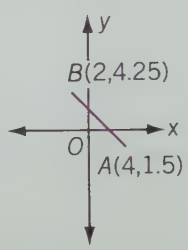


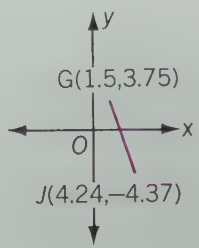


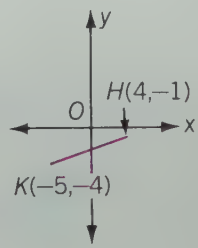
2. Find the coordinates of the midpoint of each segment.

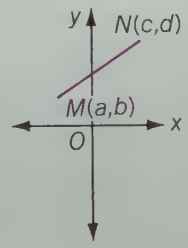












3. Find the midpoint of the segment determined by each pair of endpoints indicated.
   (a) $P(3,-2)$, $Q(-5,-3)$
   (b) $R(1.62, 8.91)$, $S(4.22, -7.05)$
   (c) $V(1.77, -1.77)$, $W(-2.99, -2.99)$
   (d) $X(a,b)$, $V(b,a)$
   (e) $P(a,b)$, $Q(e,d)$

4. A parallelogram has the vertices $A(-2,-3)$, $B(0,3)$, $C(6,5)$ and $D(4,-1)$. Find the midpoint of the diagonal $\overline{AC}$ and of the diagonal $\overline{BD}$. What relation do you notice? Do you think this relation holds for all parallelograms? Test your hypothesis with (1) a rectangle; (2) a square; (3) an oblique parallelogram; each of your own choice drawn on graph paper. Write this relation as a rule.

5. Find the midpoint of each side of the triangle ABC in which the vertices are $A(1,6)$, $B(3,2)$, $C(-2,-2)$. Draw triangle ABC on graph paper.
   Then join each vertex with the midpoint of the side opposite. What relation do you notice? Test other triangles to see if this relation holds.

7·5 Collinear Points: Method 1

Study Figure 7-6 in which $\overline{AM}$ is a straight line. What can you say about the slope of each segment of the line? *Hint!* Compare the slopes of individual segments, i.e., $\overline{AB}$, $\overline{CD}$, etc.

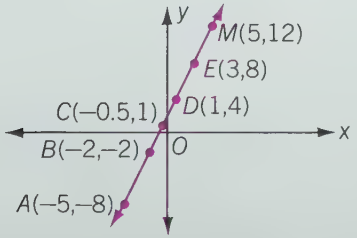


Figure 7-6

The slope of $\overline{AM}$ is 2; the slope of $\overline{DE}$ is 2; the slope of $\overline{BD}$ is 2; the slope of any segment that is a subset of $\overline{AM}$ is 2.

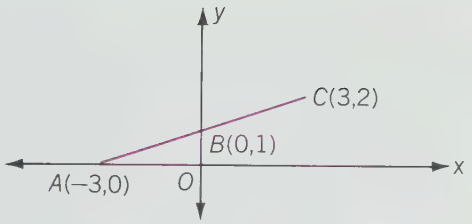


Figure 7-7

Refer to Figure 7-7. In this case you are given the points A, B and C. What can you say about the geometric figure ABC? Is it part of a straight line or do $\overline{BA}$ and $\overline{BC}$ form an angle less than 180°?

The slope of $\overline{AB}$ is $\frac{1}{3}$. The slope of $\overline{BC}$ is $\frac{1}{3}$. The intersection of $\overline{AB}$ and $\overline{BC}$ is the point B.
From the two facts:

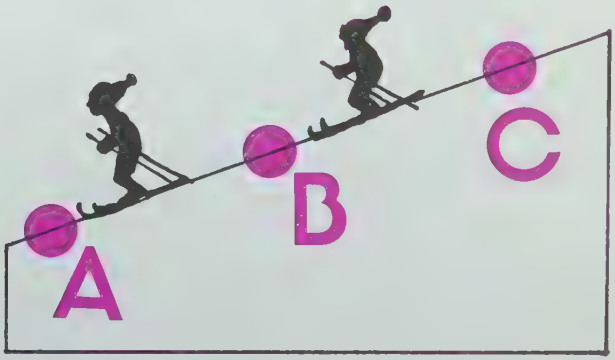
(a) the intersection of the two segments is at least one point;

(b) the slope of the first segment is equal to the slope of the second segment;

we are able to conclude that $\overline{AB}$ and $\overline{BC}$ form part of a straight line. Hence, we say points A, B, and C are *collinear*. Three points are said to be collinear if they are contained in the same (straight) line.

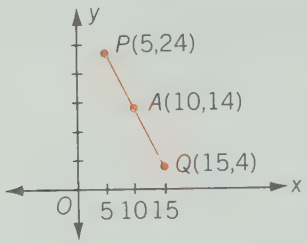
In analytical geometry, three points are said to be collinear if the three points are subsets of a line. We have just seen how we can show that three points are subsets of a line.

> Three points, A, B, C, are collinear if slope of $\overline{AB}$ = slope of of $\overline{BC}$.



Example 1

Determine whether or not points A,P,Q, are collinear given $A(10,14)$, $P(5,24)$, $Q(15,4)$.



Solution

While it is not necessary to represent this graphically, it will help you to visualize the problem.

Find the slope of any two segments that can be formed using 3 points. Let us use $\overline{AP}$ and $\overline{PQ}$.

$$\text{slope of } \overline{AP} = \frac{14-24}{10-5} \qquad \text{slope of } \overline{PQ} = \frac{4-24}{15-5}$$
$$= -2 \qquad = -2$$

Since the slopes of the two segments are equal, the three points are collinear.

$$m_1 = m_2$$

Example 2

Determine whether or not points $B(9,-3)$, $R(4,1)$, $Z(12,-7)$ are collinear.

Solution

Select the two segments $\overline{BR}$ and $\overline{RZ}$. Find the slope of each.

$$\text{slope of } \overline{BR} = \frac{1-(-3)}{4-9} \qquad \text{slope of } \overline{RZ} = \frac{-7-1}{12-4}$$
$$= \frac{4}{-5} \qquad = \frac{-8}{8}$$
$$= -\tfrac{4}{5} \qquad = -1$$

Since the slopes of the two segments are not equal we conclude the three points are not collinear.

EXERCISE 7·5

1. Determine whether R, S and T are subsets of the same line, given $R(-7,-5)$, $S(7,1)$ and $T(16,22)$.
2. Determine whether $K(-3,6)$, $L(1,1)$ and $N(5,8.5)$ are collinear.
3. Determine whether $F(-7,6)$, $G(-3,-1.5)$ and $H(2.5, 4.35)$ are collinear.
4. Determine whether the three points in each set are collinear.
   (a) $\{(-5,2), (4,6), (22,14)\}$
   (b) $\{M(-2,2), N(5,4), P(12,6)\}$
   (c) $\{F(3,-11), E(-3,7), J(-6,6)\}$
   (d) $\{(11,-6), (-3,6), (-6,2)\}$

7·6 Collinear Points: Method 2

You have learned that the sum of the measures of the two shortest sides of a triangle must be greater than the measure of the longest side. Let us see what conclusion can be drawn if the sum of the measures of the two shortest sides is equal to the measure of the longest.

DISCOVERY EXERCISE

Consider points A,B,C to be the vertices of a triangle. With respect to the same set of axes draw the following triangles.

(a) $A(0,0)$, $B(3,6)$, $C(8,0)$
(b) $A(0,0)$, $B(3,4)$, $C(8,0)$
(c) $A(0,0)$, $B(3,2)$, $C(8,0)$
(d) $A(0,0)$, $B(3,\frac{1}{4})$, $C(8,0)$
(e) $A(0,0)$, $B(3,0)$, $C(8,0)$

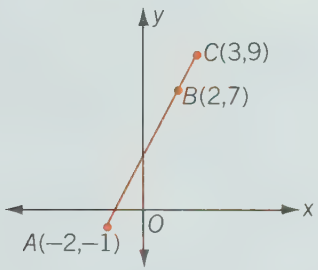
1. What do you observe happens to the triangles?
2. Complete the table.

	$m(\overline{AB})$	$m(\overline{BC})$	$m(\overline{AC})$
(a)			
(c)			
(e)			

3. Compare for each in 2: $m(\overline{AB}) + m(\overline{BC})$ with $m(\overline{AC})$.
4. Three points A, B, C, with B between A and C, are collinear if _____. (Complete this sentence with reference to question 3.)

Example 1

Determine whether the points $A(-2,-1)$, $B(2,7)$ and $C(3,9)$ are collinear.



Solution

It appears from this graph that B is between A and C so we shall see whether or not

$$m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC}).$$

$$\begin{aligned} m(\overline{AB}) &= \sqrt{(2-(-2))^2 + (7-(-1))^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$

$$\begin{aligned} m(\overline{BC}) &= \sqrt{(3-2)^2 + (9-7)^2} \\ &= \sqrt{1+4} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} m(\overline{AC}) &= \sqrt{(3-(-2))^2 + (9-(-1))^2} \\ &= \sqrt{5^2+10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

$4\sqrt{5} + \sqrt{5}$ is equal to $5\sqrt{5}$. Since $m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$ the three points A, B, and C are collinear.

Which sets of points are collinear?

EXERCISE 7·6

In each of the questions find the measure of each segment indicated then determine whether or not the 3 points given are subsets of a line.

1. $M(-4,-4)$, $N(0,-1)$, $P(12,4)$
   $\overline{MN}$, $\overline{NP}$, $\overline{MP}$
2. $A(-2,8)$, $B(2,0)$, $C(4,-4)$
   $\overline{AB}$, $\overline{BC}$, $\overline{AC}$
3. $R(1,4)$, $S(2,10)$, $T(3,0)$
   $\overline{RS}$, $\overline{ST}$, $\overline{RT}$
4. $H(5,2)$, $N(3,0)$, $G(0,-4)$
   $\overline{HN}$, $\overline{NG}$, $\overline{HG}$

Determine whether the three points in questions 5 and 6 are collinear.

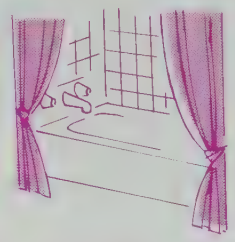
5. $Z(-2,6)$, $X(0,6)$, $Y(2,2)$
6. $F(-3,-11)$, $G(0,-2)$, $H(3,7)$
7. Three of the following four points are collinear. Determine which three are collinear.
   $(2,-\frac{23}{12})$, $(4,-\frac{7}{2})$, $(6,-\frac{63}{12})$, $(-2,\frac{17}{12})$.
8. Is the midpoint of the segment determined by the two endpoints (2,–3) and (18,9) collinear with the two points (18,1) and (2,5)?

Using the Library

1. *The Grant Golden Book of Mathematics.* Read pages 52 and 53. What does the word *analytic* mean? What is the derivation of the word? Use a dictionary.

2. *Romping Through Mathematics.* Read pages 93 and on as far as you can in that chapter. Can you solve the problem on page 96 about the sugar and soap?

Niagara PLUMBING



Our company is engaged in a wide field of services to the public. The jobs range from service calls in private homes to the installation of plumbing and heating in large hospitals or schools.

No matter what the size of job we do, we must know if our charges are adequate to cover our costs, overhead and profit.

You as a student are asked to help us, to see if we made a reasonable percentage of profit on the following job.

A bill was sent to Mr. X for remodelling work done to his home by us, the plumbing contractor. The amount of the bill was $1,927.11.

Additional information to Problem:

(a) The cost of material to the plumbing contractor was $1,175.00 less 2%.

(b) Cost of labour:
1 man per hour $3.65 plus:
10¢ per hr for welfare
6% per hr for holiday pay
$3\frac{1}{2}$% per hr for Statutory holiday pay
$2\frac{1}{2}$% per hr for Education Fund
3% per hr for Industrial

Total hours on job: 1 man, 43 hours.

(c) The electrician on the job billed us and we included his bill in ours. The electrical bill was $225.00 and we added 5% for handling.

(d) Permits were taken out for $20.00 and included in the bill.

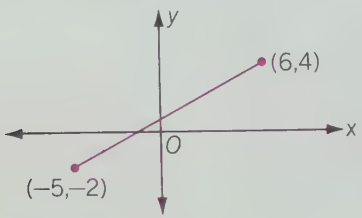
(e) Overhead charged: 10% of total cost.

1. Find the total cost per man per hour.
2. Find the total cost of the job to the plumbing contractor.
3. What was the percentage of profit on the job?

REVIEW EXERCISE 7

1. Find the measure of the segment determined by the pair of endpoints whose coordinates are
   (a) (6,9), (11,13) (b) (12,4), (8,10)
   (c) (−7,6), (2,−3)

2. Find the coordinates of the midpoints of each segment:
   (a)



   (b) (3.2,−4.6), (−4.6,6.2)

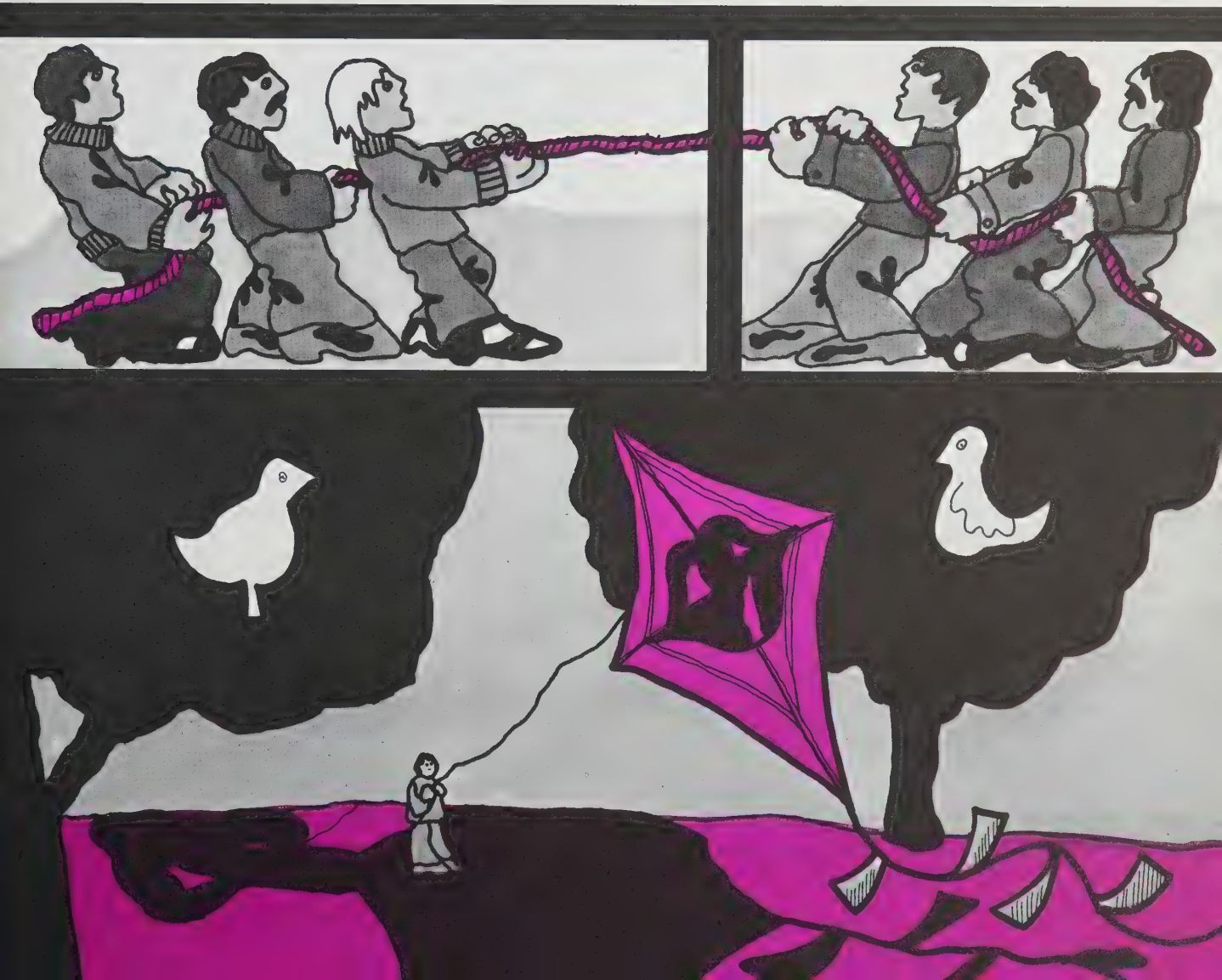
3. Determine whether the three points in each set are collinear.
   (a) {(6,4), (−6,−4), (3,2)}
   (b) {(−4,5), (−1,9), (2,13)}

4. Write a flow chart for finding the midpoint of a segment given the coordinates of the endpoints.

5. A circle has a diameter with endpoints $G(-8,6)$ and $H(6,-2)$. What are the coordinates of the centre of the circle? Find the coordinates of any four other points on the circle.

6. If you are given two segments, is it sufficient to show that the slopes are equal in order to conclude the two segments are subsets of the same line? Explain.

Chapter Eight
VECTORS



8·1 Directed Segments

Let us consider displacements (movements) from one point to another in the Cartesian plane. Our starting point will be the origin. If we move horizontally right 3 units, then move vertically up 4 units, we arrive at the point (3,4). We could have moved directly from the origin to the point (3,4). The single displacement from the origin to (3,4) ends in the same position as the two displacements do.

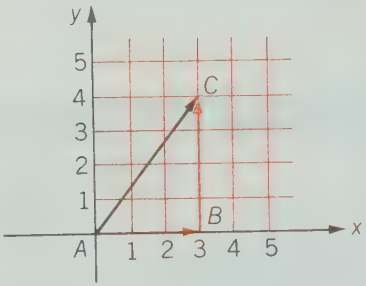


Figure 8-1

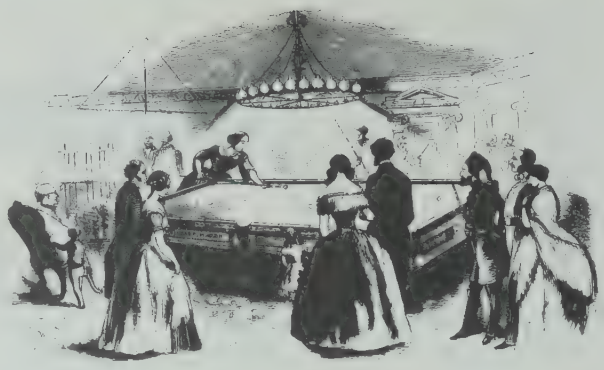
In geometry we indicate displacements by *directed segments*. We indicate each displacement by an arrow with the tail at the *initial point* and the head at the *terminal point*. In Figure 8-1 the three directed segments of the previous paragraph are shown.

> Graphically, a directed segment is represented by an arrow: the tail at the initial point and the head at the terminal point.

Algebraically, we can indicate the first movement from (0,0) to (3,0) by [(0,0), (3,0)]; the second displacement from (3,0) to (3,4) by [(3,0), (3,4)]; the single displacement from (0,0) to (3,4) by [(0,0), (3,4)].

Notice the directed segment is indicated by an ordered pair of ordered pairs in a set of square brackets. If we label the three points A, B, C as shown in Figure 8-1 we may indicate the directed segments by $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively.

> The square brackets [ ] containing an ordered pair indicate a directed line segment.



There are many reasons why a billiard ball, when struck, goes where it does. The mathematics of vectors can help to analyse them.

> The symbol $\rightarrow$ used over the names of two points, i.e., $\overrightarrow{AB}$, indicates the directed segment. The first letter indicates the initial point and the second letter indicates the terminal point.

Example 1

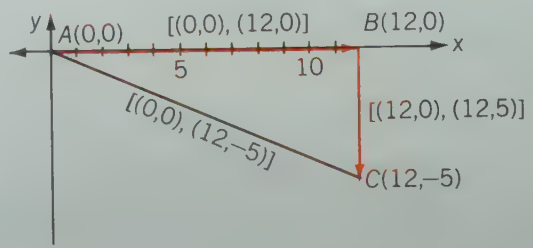
(a) Indicate on graph paper a displacement of 12 units right horizontally from the origin and 5 units vertically down. Name the initial point A, the terminal point of the first displacement B, the final terminal point C.

(b) Write each displacement and the single displacement that produces the same result as the two in notation using

(i) square brackets, [ ], (ii) $\rightarrow$

Solution

(a)



(b) (i) First displacement: $[(0,0), (12,0)]$
Second displacement: $[(12,0), (12,-5)]$
Single displacement: $[(0,0), (12,-5)]$

(ii) First displacement: $\overrightarrow{AB}$
Second displacement: $\overrightarrow{BC}$
Single displacement: $\overrightarrow{AC}$

The directed segment $[(0,0), (12,-5)]$ is usually written $[12,-5]$ with the understanding that the initial point is $(0,0)$. *Any directed segment whose initial point is (0,0) is in standard position.* $[12,-5]$ *is the basic ordered pair and represents the basic directed segment.* We think of $[12,-5]$ as that single displacement which gives the same result as two distinct displacements $[12,0]$ and $[0,-5]$.

Consider the directed segment $[5,6]$; this is shown in Figure 8-2 by $\overrightarrow{AB}$. Other directed segments showing a movement 5 units, horizontally and 6 units upward are shown by $\overrightarrow{CD}, \overrightarrow{EF}, \overrightarrow{GH}$. The four directed segments have the same magnitude and direction but have different initial and terminal points.

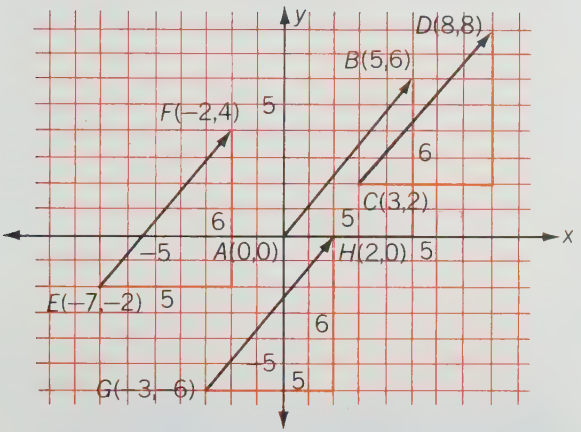


Figure 8-2

Since each directed line segment is 5 right and 6 up, or $[5,6]$, then

$$\overrightarrow{AB} = [(0,0), (5,6)] = [5,6]$$
$$\overrightarrow{CD} = [(3,2), (8,8)] = [5,6]$$
$$\overrightarrow{EF} = [(-7,-2), (-2,4)] = [5,6]$$
$$\overrightarrow{GH} = [(-3,-6), (2,0)] = [5,6]$$

Directed segments which have the same magnitude and direction are said to be equivalent.

Example 2

Given the directed segment $[(2,3), (-4,-5)]$:

(a) represent it graphically

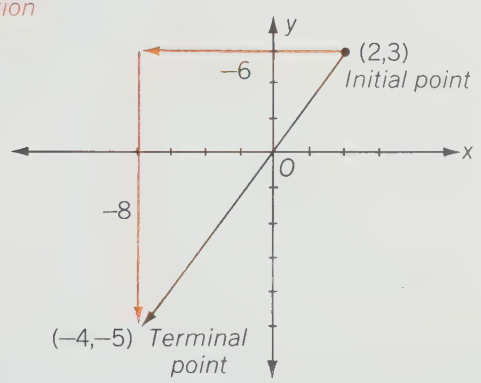
(b) represent an equivalent directed segment in standard position

(c) show graphically two other equivalent directed segments

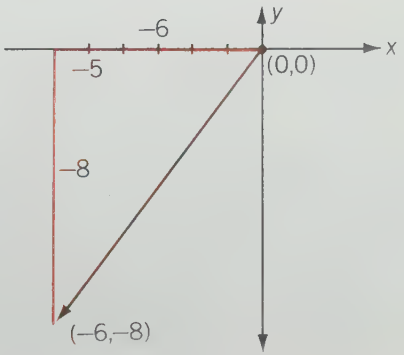
(d) find the magnitude of the original directed segment.

Solution

(a)



(b) This directed segment can be considered as a displacement of -6 units right and -8 units up. Hence, it may be written in its equivalent form $[(0,0), (-6,-8)]$ or its equivalent basic form $[-6,-8]$.

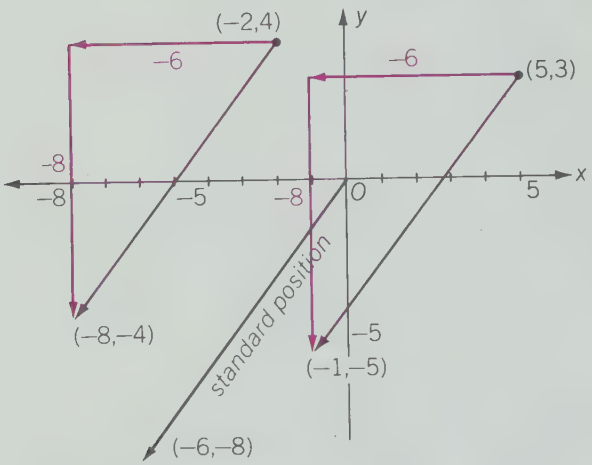


(c) Two other equivalent directed segments are found by:

choosing any point;
moving 6 units left and 8 units down.

(i) Choose (5,3).
Move 6 left and 8 down. We get (−1,−5).
The directed segment is [(5,3), (−1,−5)].

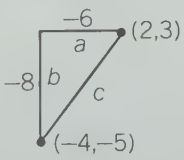
(ii) Choose (−2,4).
Move 6 left and 8 down. We get (−8,−4).
The directed segment is [(−2,4), (−8,−4)].



Notice all the equivalent directed segments are parallel to and equal in magnitude to the original directed segment and to each other.

(d) Using the Pythagorean property we have

$$a^2 + b^2 = c^2$$
$$|-6|^2 + |-8|^2 = c^2$$
$$c^2 = 100$$
$$c = 10$$



The magnitude of the directed segment is 10.

EXERCISE 8·1

Show graphically the following displacements:

1. [(4,−3), (−3,5)]
2. [(−4,−3), (4,4)]
3. [(2,3), (5,−5)]
4. [(4,1), (0,−3)].

Vectors at work in a physics lab experiment.

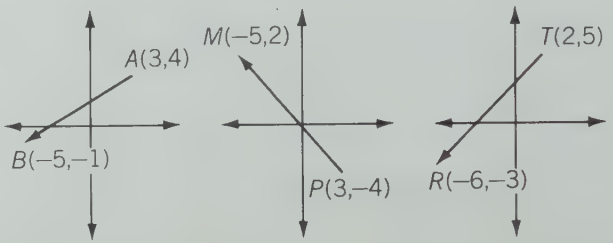
5. Write each displacement in questions 1 through 4 in equivalent algebraic form when the initial point is the origin (0,0).

6. Show graphically the standard position of each directed segment of question 5.

For each of the following write algebraically the equivalent directed segment with initial point (3,2).

7. (a) [4,3] (b) [−3,5] (c) [−4,−6]

8. Represent by the arrow notation (i.e., $\overrightarrow{AB}$) the following:



9. For each of the directed segments in question 8 write the
   (a) ordered pair notation
   (b) the basic ordered pair notation.

10. Use the Pythagorean property to find the magnitude of each directed segment.
    (a) [5,12] (b) [8,12] (c) [3,9]
    (d) [−4,10] (e) [−5,−8]

8·2 Vectors

In Section 8·1 we dealt with movements called displacements. Each displacement was described by its magnitude and its direction. Any quantity which is or can be described by a magnitude and a direction is called a *vector*. We may speak of a force of 10 pounds acting upward at an angle of 30° to the vertical (Figure 8-3) or we may speak of the speed of an airplane travelling 400 miles per hour 45° west of north (Figure 8-4). For each, a directed segment could be drawn to a specified scale so that subsequent data could be recorded in proper scale on each graphical representation.

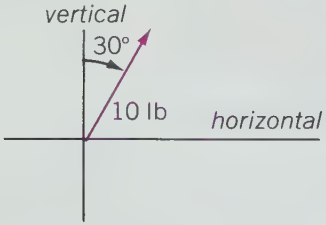


Figure 8-3

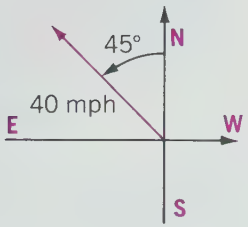


Figure 8-4

A vector is to be distinguished from a *scalar*. The scalar quantities, 5 pounds of nails, 3 cars, 13 miles, etc., state quantity or magnitude only.

> A vector quantity states magnitude and direction.
> A scalar quantity states magnitude only.

Vector quantities are represented by directed segments. Mathematicians have agreed the symbol for a vector is $\overrightarrow{AB}$. Notice this is the same as the symbol for ray. You will have to obtain from the context whether this symbol means ray or vector. In this chapter it is always vector.

Example 1

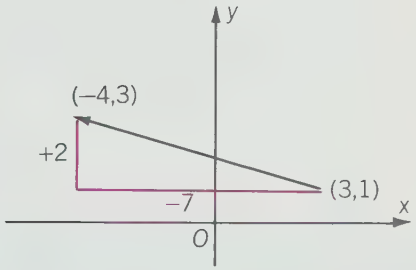
Show geometrically the following vectors.

(a) Initial point (3,1), terminal point (−4,3).

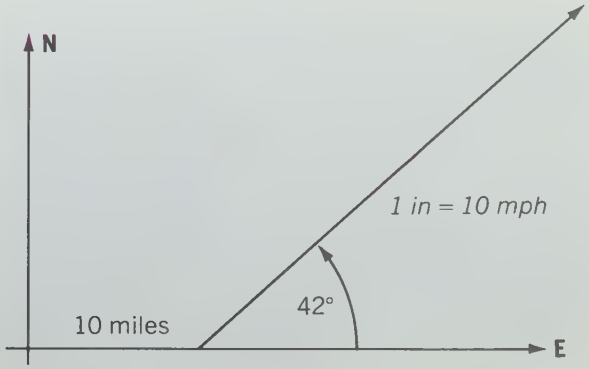
(b) Velocity of 30 miles per hour at a direction of 42° to the north of east with the initial starting point 10 miles east of Saskatoon.

Solution

(a) The vector is [(3,1), (−4,3)]. In basic form it is [−7,2].



(b) We shall interpret the *x*-axis as the east-west line (east being the positive direction on the *x*-axis) and the *y*-axis as the north-south line (north being the positive direction on the *y*-axis). We shall let the origin be Saskatoon. Scale: 1 inch = 10 miles per hour. Notice how we have shown both distance and speed on the graph.



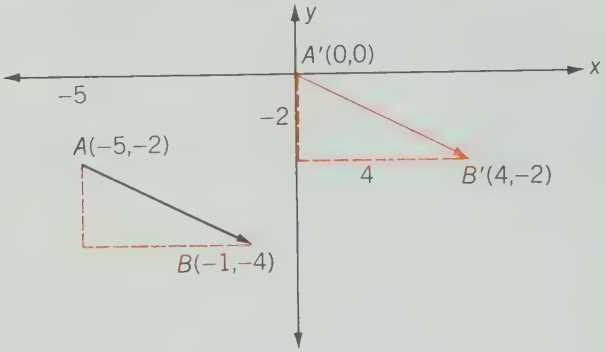
Example 2

Show geometrically the following vector.

$\overrightarrow{AB}$ where $A = (-5,-2)$ and $B = (-1,-4)$. Draw $\overrightarrow{AB}$ in standard position.

Solution

$A(-5,-2)$ is the initial point. $B(-1,-4)$ is the terminal point. Notice $\overrightarrow{A'B'}$ is equivalent to $\overrightarrow{AB}$. For simplicity when two directed segments are equivalent we shall say the vectors they represent are equal. Therefore $\overrightarrow{A'B'} = \overrightarrow{AB}$.



Two vectors are equal if and only if they have equal magnitude and the same direction.

Vector Addition

Vectors which can be represented by collinear directed segments can be added easily. Consider a displacement 3 blocks due east followed by another displacement 4 blocks due east. This is shown in Figure 8-5.

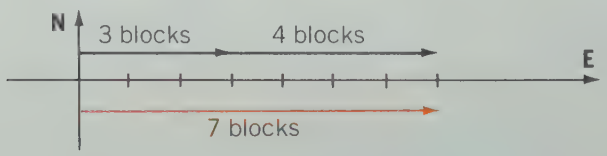


Figure 8-5

Similarly, a walk 3 blocks east followed by a walk 4 blocks west results in a final position of 1 block west of the initial point. Because eastward is considered positive, westward is negative.

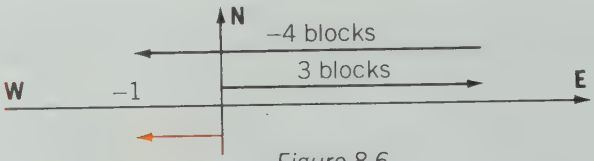


Figure 8-6

These two instances can be written in algebraic form:

3 blocks east + 4 blocks east: $[3,0] + [4,0] = [7,0]$

3 blocks east + 4 blocks west: $[3,0] + [-4,0] = [-1,0]$.

Notice in each of the two diagrams the colored directed segment indicates the one vector that is equivalent to the two vectors. We say this colored vector is the *sum* of the other two vectors.

To show geometrically the sum of two or more vectors, we draw the first vector then use the head of this vector as the initial point of the second vector. The head of the second becomes the initial point of the third, and so on. The head of the final vector is the terminal point of the sum of all the vectors.

Noncollinear vectors are added in the same way. Consider the airplane that flies 400 miles due east then 300 miles due north of Diamond City. It then flies 250 miles due west. What is the sum of these three vectors? Let us work with the first two flights. We can represent the first flight by the vector [400,0]; it indicates 400 miles east and 0 miles north. The second flight is indicated by its equivalent vector [0,300]. Graphically, the two flights are shown in Figure 8-7: $\overrightarrow{AB} = [400,0]$, $\overrightarrow{BC} = [0,300]$. The sum of the two vectors is represented by the vector $\overrightarrow{AC} = [400,300]$.

$$[400,0] + [0,300] = [400,300].$$

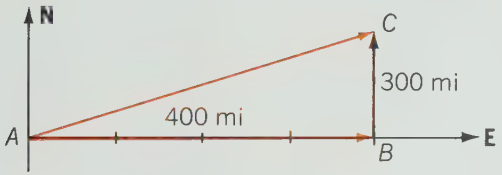


Figure 8-7

Let us now work with the three flights. The third flight is represented by the vector [–250,0]. We can add three vectors in the manner illustrated above. Addition of vectors is a binary function: it is performed on two vectors at one time. The sum of the first two vectors is then added to the third vector.

Geometrically the first vector is drawn then the second is drawn from the terminal point of the first, then the third is drawn from the terminal point of the second. The third terminal point is (150,300).

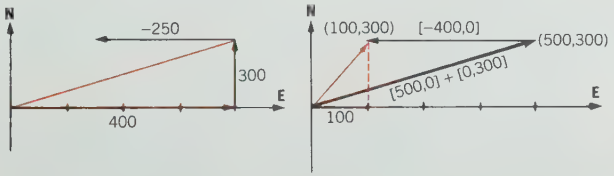


Figure 8-8

Algebraically, the addition is written in two steps:

$$[400,0] + [0,300] + [-250,0] = [400,300] + [-250,0]$$
$$= [150,300]$$

The one vector which is equivalent to the three vectors is [150,300].

> When two vectors [*a*,*b*] and [*c*,*d*] are added the result is a third vector [*a* + *c*, *b* + *d*]

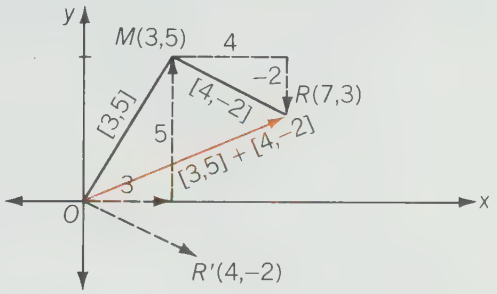
Example 3

Find the vector sum of [3,5] and [4,–2].

Solution

[3,5] represents a trip 3 units to the right of the origin and 5 units up to the point *M*(3,5). To this we add [4,–2] by drawing a vector equivalent to $\overrightarrow{OR}$ ([4,–2]) but with the tail at *M*(3,5). We need only move 4 units to the right of *M* and 2 units down. The sum is the vector shown in color [7,3].

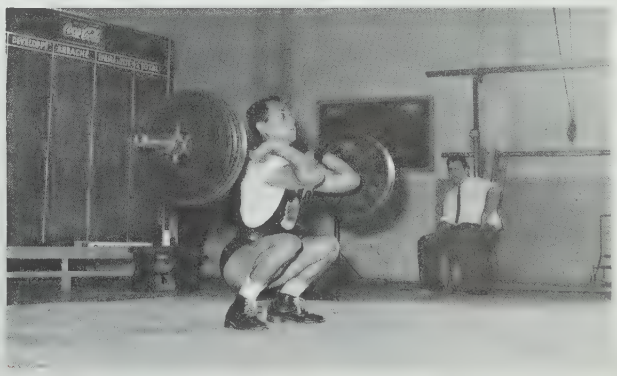
Hence, $[3,5] + [4,-2] = [3 + 4, 5 - 2]$
$= [7,3]$



EXERCISE 8·2

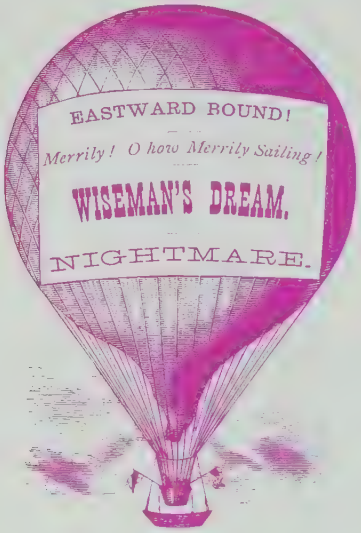
Show graphically the following vectors. Explain the meaning of the axes of the Cartesian plane for each.

1. A distance of 500 miles due south.
2. A velocity of 10 miles per hour due west.
3. An upward force of 60 pounds acting 30° to the vertical.
4. An upward force of 15 pounds vertically.



Graph the following sets of vectors. Show by a colored directed segment the one vector which represents the sum of each set.

5. A force of 450 pounds upward opposed by a force downward of 10 pounds.



6. 20 miles due south followed by a movement 35 miles due north.
7. A displacement of 30 miles due east followed by one 40 miles due north.
8. A velocity of 5 miles per hour south and 12 miles per hour west.
9. The following movements: 2 blocks west, 3 blocks south, 4 blocks east and 2 blocks north.

Express the vector $\overrightarrow{PQ}$ as a basic ordered pair.

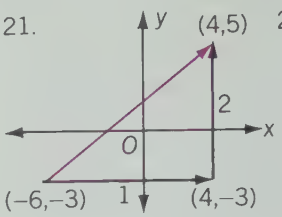
10. $P(3,2)$, $Q(5,-7)$
11. $P(7,9)$, $Q(-3,-7)$
12. $P(-6,-3)$, $Q(-2,7)$
13. $Q(7,3)$, $P(7,6)$
14. $Q(6,-2)$, $P(-6,-2)$

Find the sum of vectors (a) algebraically, (b) geometrically.

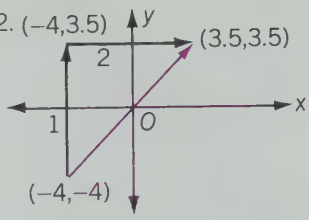
15. $[4,7] + [3,2]$
16. $[5,-2] + [-6,8]$
17. $[7,-6] + [-6,-7]$
18. $[-3,2] + [3,-2]$
19. $[3,2] + [-5,6] + [6,-3]$
20. $[4,5] + [4,5] + [4,5]$

Write the algebraic statements for the vector addition illustrated in each diagram.

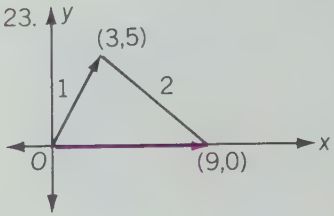
21.



22. $(-4,3.5)$



23.



8·3 Properties of Vector Operations

DISCOVERY EXERCISES

Calculate the sum of each:

1. (a) $[8,7] + [3,-5]$ (b) $[3,-5] + [8,7]$
2. (a) $[6,-3] + [-2,5]$ (b) $[-2,5] + [6,-3]$
3. (a) $[16,-21] + [9,-14]$ (b) $[9,-14] + [16,-21]$.
4. Compare the answers of each pair of sums in questions 1 through 3. What do you observe? The three questions illustrate a property of vector addition. State, in your own words, the property illustrated.
5. The fact that the sum of two vectors is always another vector illustrates the property of closure under addition in the set of vectors. Is the following an exception to this rule?

$$[3,14] + [-3,-14] = [0,0]$$

Is $[0,0]$ a vector? Explain the vector $[0,0]$ in terms of trips in a Cartesian plane.

Calculate the sum of each:

6. (a) $[3,7] + [0,0]$ (b) $[0,0] + [3,7]$
7. (a) $[-5,-6] + [0,0]$ (b) $[0,0] + [-5,-6]$
8. (a) $[17,-23] + [0,0]$ (b) $[0,0] + [17,-23]$
9. Referring to questions 6 through 8 we notice that $[0,0]$ in vector addition acts just like 0 in addition in the set of real numbers. In the set of real numbers $3\frac{1}{2} + 0 = 0 + 3\frac{1}{2} = 3\frac{1}{2}$. State in your own words the additive identity property of vector addition.

Calculate the sum of each:

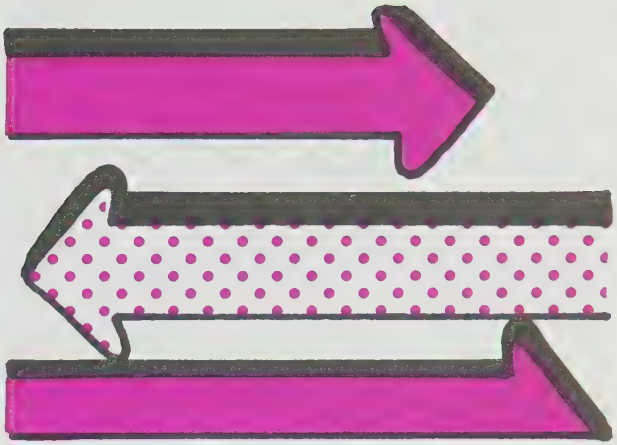
10. $[3,5] + [-3,-5]$ 11. $[-8,-7] + [8,7]$
12. $[-11,15] + [11,-15]$ 13. $[35,-6] + [-35,6]$
14. $[x,y] + [-x,-y]$, $x,y \in R$
15. $[a,b] + [-a,-b]$, $a, b \in R$
16. What do you notice about the sum in each of questions 10 through 15?
17. In the set of real numbers $7 + (-7) = 0$, $-55 + 55 = 0$. If $a \in R$ and $a + (-a) = 0$ then $-a$ is the additive inverse of a and a is the additive inverse of $-a$. Now refer to questions 10 through 15 and state, in your own words, the rule illustrated.

Give the additive inverse (commonly called *inverse*) for each vector.

18. $[62,27]$ 19. $[-41,16]$
20. $[25,-6]$ 21. $[-21,-12]$

Calculate the sum of each:

22. (a) $([13,4] + [7,11]) + [-2,-5]$
    (b) $[13,4] + ([7,11] + [-2,-5])$
23. (a) $([-4,-9] + [11,-2]) + [7,6]$
    (b) $[-4,-9] + ([11,-2] + [7,6])$
24. (a) $([29,-3] + [-4,11]) + [16,-5]$
    (b) $[29,-3] + ([-4,11] + [16,-5])$



25. What property do questions 22 through 24 illustrate? State in your own words this property in full.
26. In the set of real numbers we define subtraction in terms of addition as follows:

$$23 - 47 = 23 + (-47); \quad 35 - (-7) = 35 + (-[-7])$$
$$= -24 \qquad\qquad = 35 + (7)$$
$$= 42$$

    State this rule in your own words.
27. In vectors, subtraction is defined in a similar manner.

$$[14,-16] - [-31,6] = [14,-16] + [-(-31,6)]$$
$$= [14,-16] + [31,-6]$$
$$= [14 + 31, -16 - 6]$$
$$= [45,-22]$$

    Can you think of another way to get the answer? Explain your method.

    Perform the subtraction in each:

    (a) $[16,-4] - [7,-8]$

    (b) $[-23, 19] - [12,-4]$

    (c) $[127,-529] - [-227, 61]$

    (d) $[3.14,-6.15] - [-6.41,-9.32]$
28. State in your own words the rule for subtraction in the set of vectors.
29. Devise a scheme to illustrate geometrically subtraction in the set of vectors.

30. A vector is sometimes indicated by an arrow over a small case letter:
$\vec{a}, \vec{m}, \vec{g}$. If $\vec{a} = [13,-11]$, $\vec{c} = [29,-31]$, and $d = [-6,17]$ find

(a) the inverse of $\vec{a}$ (b) $\vec{c} - \vec{a}$

(c) $(\vec{a} + \vec{c}) + \vec{d}$ (d) $\vec{a} + (\vec{c} + \vec{d})$.

31. If $\vec{m} = [3,-4]$, $\vec{r} = [5,-2]$, $\vec{s} = [-6,-3]$ calculate

(a) $\vec{m} + \vec{m}$ (b) $\vec{m} + \vec{m} + \vec{m}$

(c) $\vec{m} + \vec{m} + \vec{m} + \vec{m}$

(d) $\vec{r} + \vec{r} + \vec{r} + \vec{r} + \vec{r} + \vec{r} + \vec{r}$

(e) $\vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s}$

(f) $\vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s} + \vec{s}$

32. Can you devise an easy way to evaluate questions like those in question 31? When you have a method you think will work evaluate each of the following, given $f = [-5,4]$.

(a) $\vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f} + \vec{f}$

(b) $\vec{f} + \vec{f}$

33. Successive addition of the same vector can be simplified by *scalar multiplication of vectors.*

$$\vec{a} + \vec{a} + \vec{a} = 3\vec{a}$$

If $\vec{a} = [5,-2]$ then

$[5,-2] + [5,-2] + [5,-2] = [10,-4] + [5,-2] = [15,-6]$

$3[5,-2] = [3 \times 5, 3 \times (-2)] = [15,-6]$

Using the same pattern

$[40,3] + [40,3] + [40,3] + [40,3] + [40,3] + [40,3] + [40,3] + [40,3] + [40,3] + [40,3] = 10[40,3] = [10 \times 40, 10 \times 3] = [400,30]$

Find the products of each, given

$$\vec{g} = [7,-6], \quad \vec{k} = [-13,3.5].$$

(a) $3\vec{g}$ (b) $4\vec{k}$ (c) $12\vec{g}$ (d) $15\vec{k}$

(e) $4\vec{g} + 9\vec{k}$ (f) $-6\vec{g} + \frac{3}{2}\vec{k} - \frac{3}{4}\vec{g}$

34. If $a, b, c \in R$ we may write $c(a + b) = ca + cb$. What is this property called in the set of real numbers?

Find the products and sums in each, given

$$\vec{m} = [-3,7], \quad \vec{n} = [8,-9].$$

35. (a) $2\vec{m} + 2\vec{n}$ (b) $2(\vec{m} + \vec{n})$

36. (a) $3\vec{m} + 3\vec{n}$ (b) $3(\vec{m} + \vec{n})$

37. (a) $7\vec{m} + 7\vec{n}$ (b) $7(\vec{m} + \vec{n})$

38. What do you notice about the answers to questions 35 through 37? The six questions illustrate a property of *scalar multiplication over vector addition.* State this property in your own words.

39. The rules used in solving for variables in the set of real numbers are restated and used in solving for unknown vectors.

Solve for $\vec{x}$ in each case. Complete the solution in (b).

(a) $\vec{x} + \vec{y} = \vec{z}$

Subtract $\vec{y}$ from each side.

$\vec{x} + \vec{y} - \vec{y} = \vec{z} - \vec{y}$

Zero vector $\vec{x} + \vec{0} = \vec{z} - \vec{y}$

$\vec{x} = \vec{z} - \vec{y}$

(b) $2(\vec{x} - \vec{y}) = 3(\vec{x} + \vec{z})$

Scalar multiplication is distributive over addition and subtraction in vectors.

$2\vec{x} - 2\vec{y} = \ ?$

$2\vec{x} - 2\vec{y} - 3\vec{x} + 2\vec{y} = 3\vec{x} - 3\vec{x} + 3\vec{z} + 2\vec{y}$

Why?

$-\vec{x} = \ ?$

$(-1)(-\vec{x}) = -1(?)$

Scalar multiplicative property of equality of vectors.

$\vec{x} = ?$

40. Solve for x in each vector equation.
   (a) $2\vec{x} + 3\vec{y} = 3\vec{z}$
   (b) $3\vec{y} + 4\vec{x} = 6\vec{x} - 6\vec{z}$
   (c) $5\vec{x} - 7\vec{y} = 3\vec{y} + 2\vec{x} - 6\vec{z}$
   (d) $6(\vec{x} - \vec{y}) = 4(\vec{x} - \vec{y})$
   (e) $8(\vec{x} - \vec{y}) - 6(\vec{x} - \vec{y}) = 7\vec{y}$
   (f) $2(\vec{x} - 2\vec{y}) - 3(3\vec{x} - 2\vec{y}) = 0$
   (g) $\frac{3}{2}(\frac{1}{4}\vec{x} - \frac{1}{2}\vec{y}) - \frac{5}{4}(\frac{1}{3}\vec{x} - \frac{1}{5}\vec{y}) = \frac{3}{8}\vec{y}$
   (h) $\frac{3}{5}(\vec{x} + \frac{3}{4}\vec{y} - \vec{z}) = \frac{1}{4}(\frac{1}{5}\vec{x} - \frac{3}{5}\vec{y} - \frac{2}{3}\vec{z})$

EXERCISE 8·3

1. Show geometrically $[3,4] + [3,4]$.
   Is $[3,4] + [3,4] = 2[3,4] = [6,8]$?
2. Show geometrically that
   $$[-3,-4] + [5,8] = [5,8] + [-3,-4].$$
3. Show geometrically that
   $$([2,4] + [3,5]) + [4,3] = [2,4] + ([3,5] + [4,3]).$$
4. Show geometrically that
   $3\vec{a} + 3\vec{b} = 3(\vec{a} + \vec{b})$ where $\vec{a} = [4,3]$, $\vec{b} = [-2,-4]$.

Project

Project 1

In **Book 2,** Section 11·5, Project 1 is designed to achieve the same objective as this project. If apparatus and material are not available for this project the student and teacher should consider Project 1, if it has not been done previously.

Purpose

To find a geometric relation among three vectors, in equilibrium acting at a point.

Method

1. Set up the apparatus as shown in Figure 1. The pulleys should move freely.
2. Tap the board sharply until the system comes to rest.
3. Mark a dot on the paper under the knot A and mark the directions of $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

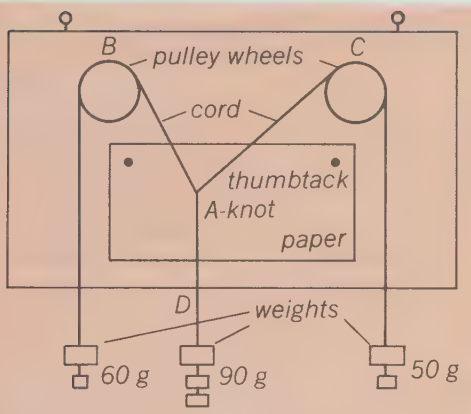


Figure 1

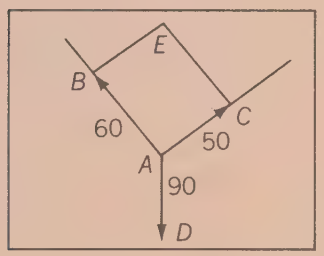


Figure 2

4. Remove the paper and draw to scale (using the magnitude) $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.
5. Using $\overrightarrow{AB}$ and $\overrightarrow{AC}$ construct a parallelogram ($ABEC$).
6. Draw and find the magnitude of $\overrightarrow{AE}$.
7. Repeat steps 1 through 6 but use any three weights so long as no one weight is greater than the sum of the other two. These results should be posted about the room so that each group can benefit from the work of the other groups.

Questions

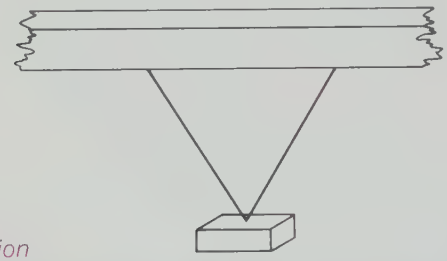
1. Compare the magnitudes of $\overrightarrow{AE}$ and $\overrightarrow{AD}$. What do you observe?
2. Compare the directions of $\overrightarrow{AE}$ and $\overrightarrow{AD}$. What do you observe?
3. What relations exist between $\overrightarrow{AE}$ and $\overrightarrow{AD}$?
4. What conclusion do you draw regarding the diagonal and the two vectors that form two sides of the parallelogram?

8·4 Scale Drawing

So far we have drawn all the vectors on grid paper. We can draw the vectors to scale on plain paper as you did in Project 1. We need not, nor, in fact, do we always give vectors in terms of the coordinates of the endpoints. The following examples will illustrate various types of problems that can be solved by scale drawings.

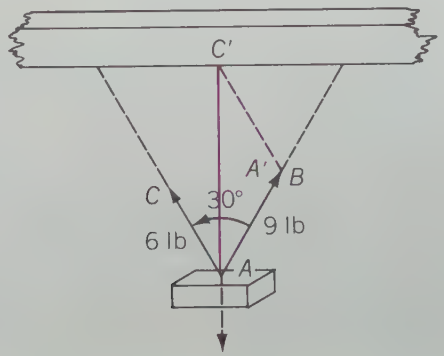
Example 1

An object is suspended from an overhead beam by two ropes as shown in the diagram. The tension in one rope is 6 pounds while the tension in the second rope is 9 pounds. The measure of the angle between the two ropes is 30°. Find the weight of the object.



Solution

We will draw two directed segments with the measure of the angle between them 30°: $m(\overline{AC}) = 6$ units, $m(\overline{AB}) = 9$ units. Now draw a vector $A'C'$ equivalent to the 6 pound vector from the head of the 9 pound vector. Draw the vector AC'. The measure of $\vec{AC'}$ represents the weight of the object. The magnitude of $\vec{AC'}$ is 13.5. Therefore, the weight of the object is 13.5 pounds.

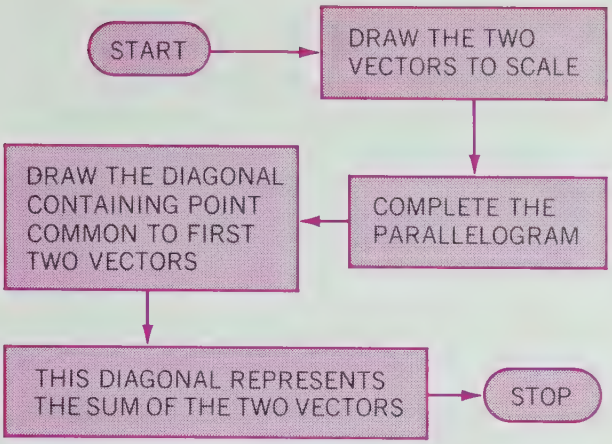


Notice that if $\vec{CC'}$ is drawn, $ACC'B$ is a parallelogram. The sum of the two vectors, $\vec{AC}$ and $\vec{AB}$, is the diagonal $\vec{AC'}$.

With reference to Example 1 these observations should be noted. Both vectors are shown starting from the same point A. In order to add vectors geometrically we said earlier that the second vector needs to start where the first vector ended. This is why we drew an equivalent or equal vector $A'C'$. We now have the first vector AB with the second vector starting at B and ending at C'. We now find the measure of the vector AC'. $\vec{AC'}$ is often called the *resultant* since we have found the result of the two vector components. The *resultant is that single vector which can take the place of two or more separate vectors. It is the sum of the vectors.*

Notice we draw a parallelogram to solve the problem in Example 1. In physics the conclusion to this type of solution is summarized in the Parallelogram of Vectors (Forces):

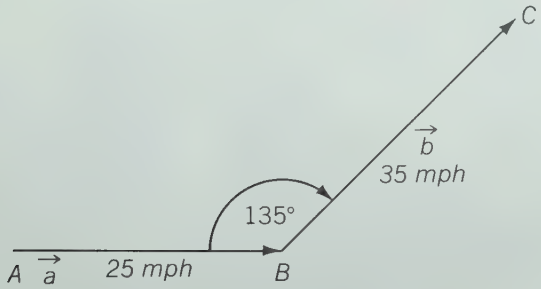
> If two vector quantities acting at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, their sum (resultant) will be represented in direction and magnitude by that diagonal of the parallelogram that passes through the same point.


START
DRAW THE TWO VECTORS TO SCALE
COMPLETE THE PARALLELOGRAM
DRAW THE DIAGONAL CONTAINING POINT COMMON TO FIRST TWO VECTORS
THIS DIAGONAL REPRESENTS THE SUM OF THE TWO VECTORS
STOP

If the problem we wish to solve can be represented by a vector diagram that involves a right triangle in which we know two sides, the third side can be solved by the Pythagorean property. If we know an acute angle and one side of the right triangle, we can use trigonometry to solve the triangle. Instances in which we use trigonometry will be illustrated in the next section. But if the triangle is not a right triangle we may be unable to solve for the unknown quantities using trigonometry. Note how we solve the problem in this example.

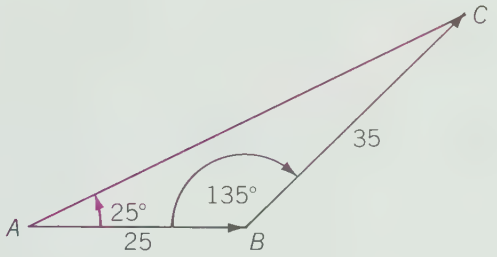
Example 2

Two vectors are shown in the diagram below. $\vec{a}$ represents 25 miles per hour and $\vec{b}$ represents 35 miles per hour. Represent to scale the vectors and find the magnitude of $\vec{AC}$ and its direction.



Solution

Choose a suitable scale. In this case 1″ = 20 miles per hour will be used. Draw vector $AB = 1\frac{1}{4}$ inches. Draw $\angle ABC$ with a measure of 135°. Now draw $\vec{BC}$ to represent 35 mph; hence, vector BC is $1\frac{3}{4}$ inches. This diagram is a scale drawing of the vector system. We now draw $\vec{AC}$. Using the scale we find $\vec{AC}$ represents 60 miles per hour. Its direction forms an angle of 25° with the 25 mph vector.



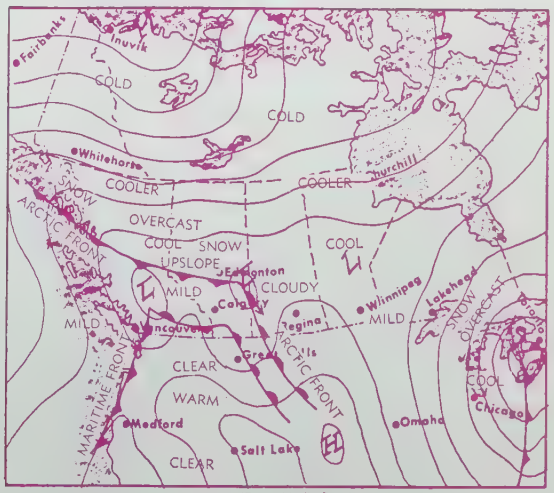
Notice that the final figure we used is a triangle. This method of adding vectors is known as the *Triangle Method of Addition of Vectors*

(a) Draw a vector to scale.

(b) At its terminal point draw the second vector.

(c) Draw a third vector from the initial point of the first vector to the terminal point of the second vector. The third vector is the sum of the two vectors.

This method is used when the second vector originates at the terminal point of the first.

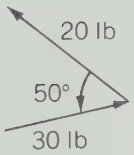


Vectors in the form of air currents determine the weather to a large extent.

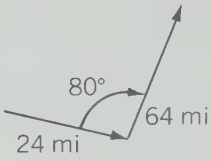
EXERCISE 8·4

Draw to scale and find the sum of each vector system shown:

1.



2.



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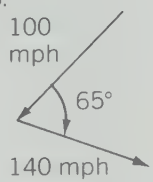

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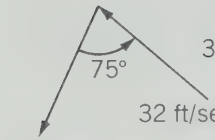
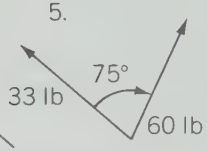
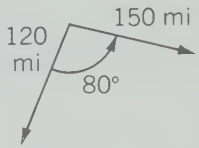


5.



6.



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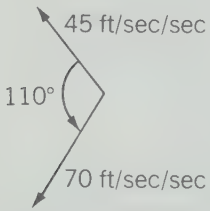


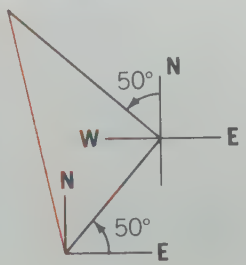
8.

9. $m(\overline{AB}) = 5$, $m(\overline{BC}) = 8$, $m\angle ABC = 40°$. Find $\overrightarrow{AC}$ and $m\angle CAB$.

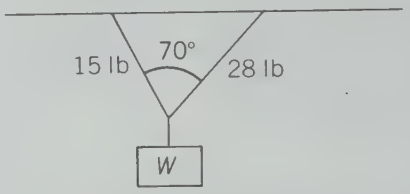
10. $m(\overline{MN}) = 13$, $m(\overline{NP}) = 24$, $m\angle MNP = 155°$. Find $\overrightarrow{MP}$ and $m\angle PMN$.

11. $m(\overline{HG}) = 8$, $m(\overline{GJ}) = 12$, $m(\overline{JK}) = 9$, $m\angle HGJ = 80°$, and $m\angle GJK = 60°$. Find $\overrightarrow{HK}$. How many solutions are possible? Explain.

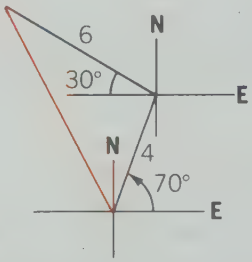
12. A trip of 15 miles 50° north of east is followed by a trip of 25 miles 50° west of north. Construct a vector diagram to scale and find the direction and distance of the terminal point from the original point.



13. An object is suspended from an overhead beam by two ropes as shown in the diagram. The tension in one rope is 15 pounds while the tension in the second rope is 28 pounds. The angle between the two ropes is 70°. Find the weight of the object.



14. A survey crew in the wilds of northern Canada cuts and maps a line 4 miles 70° north of east of a certain survey marker. The crew then turns, cuts and maps a line 6 miles 30° north of west. A final cut is made directly back to the original survey marker. Find the magnitude of the final cut and the measure of the angle between this final cut and the first cut.

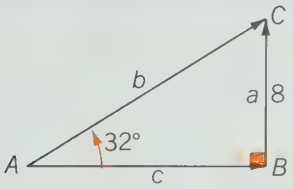


8·5 Solving Vectors by Trigonometry

This section will illustrate the use of trigonometry to solve triangles representing vectors.

Example 1

Given $m(\overline{BC}) = 8$ and $m\angle CAB = 32°$, solve the vector triangle for the two unknown vector quantities.



Solution

Recalling the trigonometric functions in Chapter 4 we write:

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\sin 32° = \frac{8}{b}.$$

Using tables we find $\sin 32° = 0.5299$.

$$0.5299 = \frac{8}{b}$$

$$b = \frac{8}{0.5299}$$

$$= 15.1 \qquad \text{(slide rule accuracy)}$$

$$m(\overline{AC}) = 15.1$$

To solve for $\overrightarrow{AB}$:

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\tan 32° = \frac{8}{c}.$$

Using tables we find $\tan 32° = 0.6249$.

$$0.6249 = \frac{8}{c}$$

$$c = \frac{8}{0.6249}$$

$$c = 12.8$$

$$m(\overline{AB}) = 12.8$$

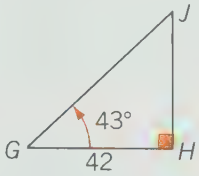
The only other unknown quantity is $m\angle ACB$. This is found by

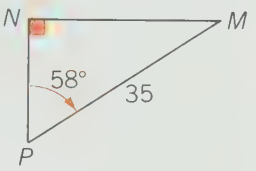
$$m\angle CAB + m\angle ACB = 90° \qquad \text{Why?}$$
$$32° + m\angle ACB = 90°$$
$$m\angle ACB = 58°$$

EXERCISE 8·5

Solve each vector triangle for the unknown vector quantities.

1. Solve for $\overrightarrow{GJ}$ and $\overrightarrow{HJ}$ given $m(\overline{GH}) = 42$ and $m\angle JGH = 43°$.

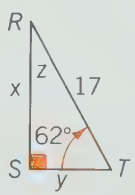




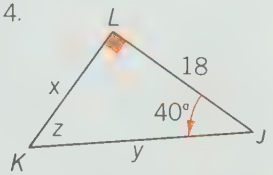
2. Solve for $\overrightarrow{MN}$ and $\overrightarrow{NP}$ given $m(\overline{MP}) = 35$ and $m\angle NPM = 58°$.

Find x, y, z.

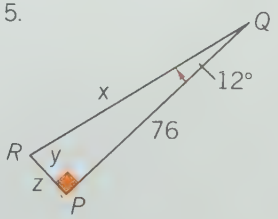
3.



4.



5.



6. An airplane flies due west from Golden at 350 miles per hour for 2 hours. It then turns due north and sometime later it crashes. By radio the pilot gives his location as 38° north of west of Golden. How far from Golden is the plane?

8·6 Application of Vectors

Vectors are used in navigation to study the movement of vessels affected by two or more velocities. We shall refer to an airplane affected by wind currents, although the same principles apply to a ship in water currents.

A pilot of an airplane sets a *course*. The course is the planned path and would be the actual path if there were no wind. Speed is also affected by the wind. *Course speed* is the speed which the plane would have in still air. When a wind blows the airplane off course the path of the airplane in the wind is the *track* and its actual speed is called *ground speed*. The navigator must make allowances for wind.

There are several ways in which surveyors or navigators indicate the direction of travel or the direction of their position relative to some point on the globe. Sections 8.4 and 8.5 used one method. Another method is illustrated here.

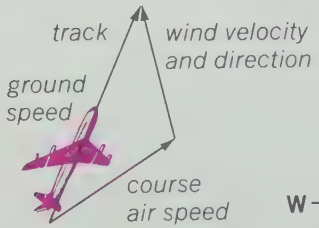


Figure 8-9

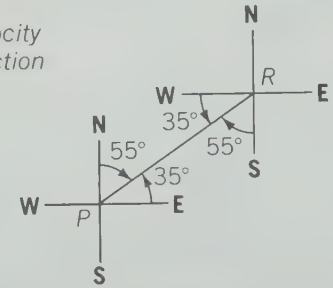
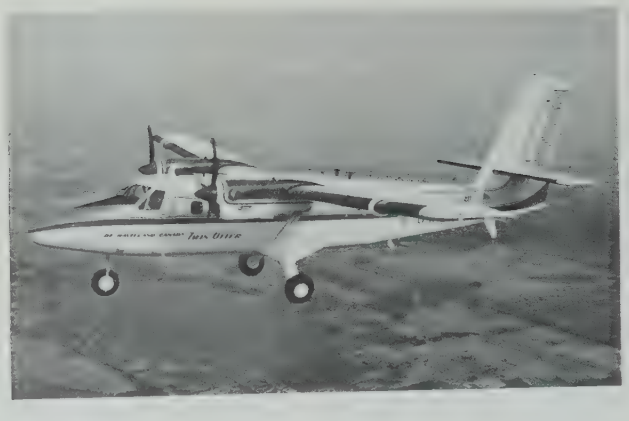


Figure 8-10

Directions or *bearings* are given relative to the north-south, east-west lines. Normally the north-south, east-west lines are oriented on the page as shown in Figure 8-10. Point R is N55°E of point P. N55°E is read *north 55° east* and means 55° east of north. Point R is E35°N of P. In turn, P is S55°W of R or W35°S of R.

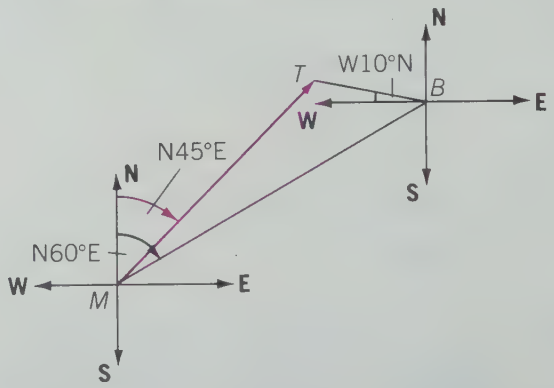
Example 1

The pilot of an airplane sets course from Montreal N60°E at an air speed of 350 miles per hour. There is a wind blowing W10°N at 90 miles per hour. Determine, by scale drawing, the track and ground speed of the aircraft.



Solution

Step 1. Draw the N-S, E-W lines and label Montreal M (the starting point) at the origin.



Step 2. Draw a vector to scale to represent 350 miles per hour and showing a bearing of N60°E. Label $\overrightarrow{MB}$.

Step 3. At B draw another set of N-S, E-W lines.

Step 4. Draw a vector from B to scale to represent 90 miles per hour showing a bearing of W10°N. ($\overrightarrow{BT}$).

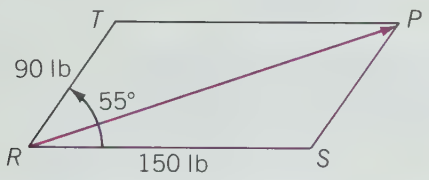
Step 5. Draw vector $\overrightarrow{MT}$.

Step 6. Measure $\angle NRT$ for the bearing. *Track is N45°E.*

Step 7. Measure $\overrightarrow{MT}$ and interpret in miles per hour. *Ground speed is 290 miles per hour.*

Example 2

Two boys pull a sled by means of two ropes which form an angle of 55°. One boy pulls with a force of 150 pounds while the second pulls with a force of 90 pounds. Find the net forward force on the sled. Show diagrammatically the direction of the force relative to the two ropes.



Solution

Step 1. Draw to scale one vector representing 150 pounds. Label $\overrightarrow{RS}$.

Step 2. Draw vector RT to scale representing 90 pounds such that $m\angle SRT = 55°$.

Step 3. Complete a parallelogram by drawing vector TP parallel to $\overrightarrow{RS}$ and vector PS parallel to $\overrightarrow{TR}$.

Step 4. Draw diagonal vector RP. Vector RP represents the net forward force on the sled. Using the scale we find $\overrightarrow{RP}$ represents approximately *190 pounds.*
The diagram is now complete.

EXERCISE 8·6

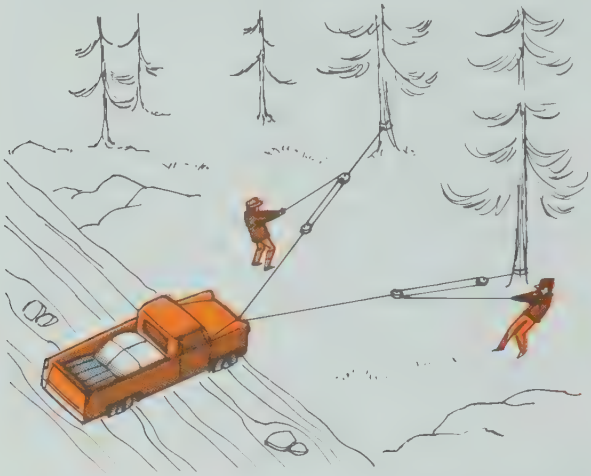
1. A pilot sets the course of his aircraft N20°W with a course speed of 190 miles per hour. He encounters a tail wind of 40 miles per hour blowing due north. Calculate the track (speed and direction) of the aircraft by scale drawing.

2. A fisherman wishes to leave the wharf and go to a certain spot in the lake which is excellent for big trout. He travels for 3 hours at 18 miles per hour N50°E. He then turns and travels 1 hour at 22 miles per hour N60°W. This route is necessary because of a neck of land jutting into the lake. How far and in what direction is the fishing spot from the wharf?

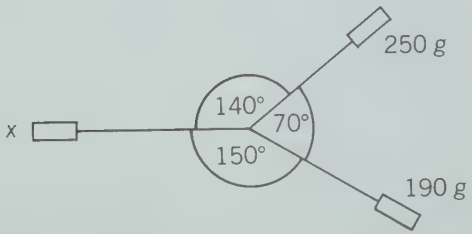
3. A sailor sets sail N30°W for 5 miles. He then tacks and travels E30°N until he is N40°E from his original position. Calculate the distance he is from his original position.

4. A jet fighter with air speed of 850 miles per hour on course S35°E encounters a jet stream blowing S50°E at 150 miles per hour. Represent this by a vector diagram and find by scale drawing the track of the jet fighter.

5. Two hunters crossing a stream with their truck got stuck. They used two block and tackle arrangements as shown below. The tensions in the cables were 550 and 400 pounds. The angle between the cables was 55°. Find the forward force on the truck.



6. Three spring balances are in equilibrium when attached at the angles shown. Find the reading on the third spring balance.



Using the Library

The Giant Golden Book of Mathematics. Read pages 59-62. Repeat both of the experiments for locating the true North. Do both results agree? Why?

Mathematics Enrichment. Read pages 4, 9, 12. Write a fictionalized account of some aspect of the life of one of these mathematicians. You may wish to do more research.

REVIEW EXERCISE 8

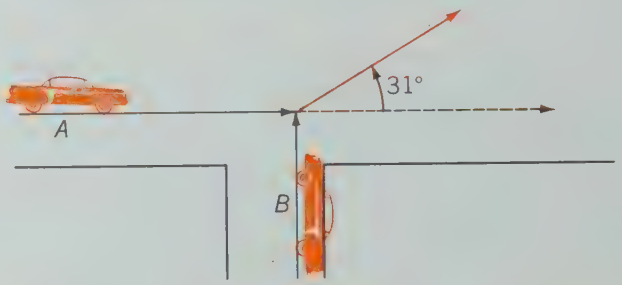
1. Calculate algebraically
   (a) $[3,5] + [-2,8] - [5,-2]$
   (b) $2[1,5] - 3[2,-3]$.

2. Show graphically each set of vectors in question 1.

3. Explain the difference between a vector quantity and a scalar quantity.

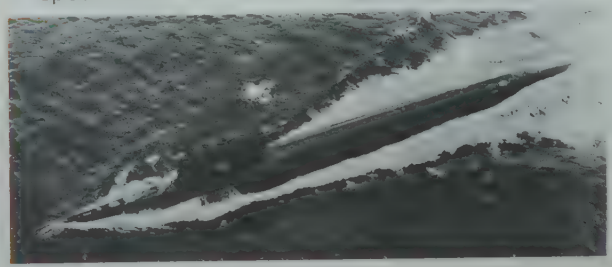
4. The commercial jet 747 sets course N45°E at 625 mph at 45,000 feet. A jet stream blowing due east at 35 mph is encountered. What can you say about the ground track (speed and direction)?

5. Two cars of the same weight collide at a city intersection as shown in the diagram. Driver of Car A said he was going 30 miles per hour and car B was going 50 miles per hour. Driver of car B said he was going 30 mph and car A was going 50 mph. Which driver was correct?



6. A battleship on course S25°W at 50 knots encounters a tide running N35°W at 12 knots. By scale drawing determine the track of the ship.

7. The track of a submarine is due west at 25 knots. A tide is running N20°W. Resolve the 25 knots into the speed of the submarine at W45°S and the speed of the tide.



Chapter Nine
PROFIT AND LOSS IN THE RETAIL BUSINESS

9·1 Operating a Business

How does a businessman determine his profit or loss? Let us consider the businessman who buys and sells articles: be they groceries, magazines, dry goods, automobiles or oil well equipment. The businessman we shall discuss usually buys a product and sells it for more than he paid for it. But does the mere fact that he sells it for more than he paid for it necessarily mean that he made a profit? How does a businessman determine for how much more than the purchase price he will sell an article?

Let us study the operation of a small clothing store to help us determine the answers to the questions posed in the preceding paragraph. The following questions will guide your discussion.

1. List all the expenses the owner of a Ladies' Wear Shop (or Men's Wear Shop) will likely incur in placing an item such as a ski jacket on the rack and offering it for sale.
2. Consider the store to have two full-time employees and to be in a small business area. Estimate a weekly cost of each expense in question 1.
3. What means does the owner have of earning enough to cover the above expenses?

Arrange to have a businessman in your neighbourhood come to your class and discuss with you the questions above, or arrange a trip to some neighbouring place of business so the owner or manager can explain to you and show you the items that go into making up the costs of operating his business.

The clothing store merchant has a cost price for each item or shipment. To this cost he adds the cost of his *expenses* and the amount he allows for his *profit*. A merchant is in business to make a profit. Without a profit a business may collapse.

(1) cost price + expenses + profit = gross cost price

Many different terms are used to mean the same thing. *Operating expenses* involved in a business, such as rent, salaries to employees, utilities, interest on invested capital, and so on, are called *overhead*. Therefore, statement 1 may be rewritten.

(2) cost price + overhead + profit = gross cost price

Can you guess another term for gross cost price? No doubt you have reasoned that

(3) cost price + overhead + profit = selling price

Sometimes business groups the two items, overhead and profit, under the one heading, *markup*. In this case, markup means the difference between cost price and selling price. Another term for markup is *margin*. Lately, the term markup has been used to mean something else. We shall discuss this other meaning later. In this section we shall use markup and margin interchangeably.

(4) $\left.\begin{matrix}\text{markup}\\ \text{margin}\end{matrix}\right\} = \text{overhead} + \text{profit}$

(5) $\text{cost price} + \left\{\begin{matrix}\text{markup}\\ \text{margin}\end{matrix}\right\} = \text{selling price}$

EXERCISE 9·1

In this chapter students are encouraged to use calculators or slide rules for computations.

1. A dealer buys a car for $300 and sells it for $325. Does he necessarily make a profit? Explain your answer.

2. List all the items you can that contribute to overhead.

3. One businessman wants a profit of at least $500 on each item or unit he sells, while another businessman is satisfied with a 3 cent profit on each item or unit. Can you explain this?

4. A merchant buys 100 articles (from a manufacturer) for $1,290. His cost of advertising this merchandise is $23.50. He calculates the cost of operating his store to sell this item is $42.35. In addition, he hires one extra clerk at a cost of $25.60. He expects a profit of $15. His advertisement reads

Does he advertise falsely? Justify your answer.

5. A manufacturer advertises in a leading national magazine. The ad uses the term *suggested retail price*. Some ads use the term *list price*. This is the price the manufacturer suggests the retailer sell the article for. Under what conditions do you think a retailer may sell the article

(a) for less? (b) for more?

HANDY COMPARISON CHART

PINTO[1]	VOLKSWAGEN[2]		MAVERICK[1]	VEGA[2]
$1919	$1985 (Super Beetle)	Price *	$2175	$2091—2-dr. sd. ($2197—2-dr. cp.)
94.0"	95.3"	Wheelbase	103"	97.0"
163"	161.8"	Overall length	179.4"	169.7"
69.4"	62.4"	Overall width	70.6"	65.4"
50.1"	59.1"	Overall height	53.0"	51.9"
55.0"/55.0"	54.3"/53.3"	Track width Front/rear	56.5"/56.5"	55.1"/54.1"
31.5'	31.5'	Turning circle	36.9'	33.0'
2030 lb.	1918 lb.	Weight (curb)	2626 lb.	2202 lb.
97.6 cu. in. 75 hp	96.6 cu. in. 60 hp	Engine size hp	170 cu. in. 100 hp	140 cu. in. 90 hp

1) Ford's suggested retail price
2) Manufacturer's suggested retail price

6. Arrange for a small committee from your class to visit some neighbourhood business. Ask the manager to estimate his annual cost due to the following overhead items: rent (or interest on capital if he owns the premises), taxes – business and property, heat, light, water and sewage, insurance, storage, shipping charges, delivery charges, sales help, office expenses, advertising, manager's salary, loss due to shoplifting or breakage, or loss or wear due to floor damage, and any others.

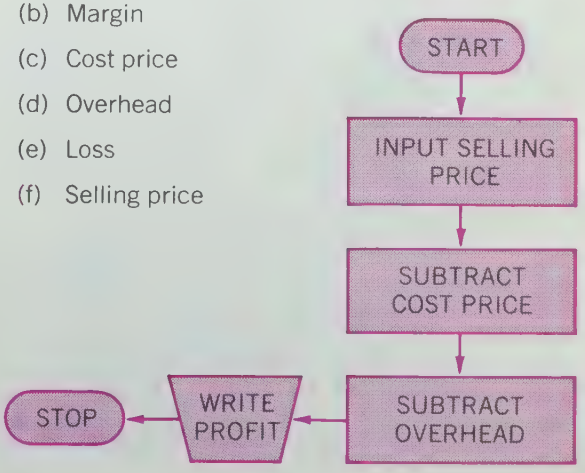
When the committee reports back to the class each student should record in some suitable way the information obtained.

7. Copy the chart and fill in as many spaces as possible.

ITEM	COST PRICE	MARGIN	GROSS PRICE	OVER-HEAD	PROFIT	LOSS	SELLING PRICE
(a)	$ 32.15	$3.50	$ –	$ 1.76	$ –	$ –	$ –
(b)	–	3.75	33.62	–	1.02	–	–
(c)	–	–	192.56	32.75	4.51	–	–
(d)	87.71	–	–	6.42	–	–	89.98
(e)	259.60	–	–	49.15	–	4.90	–
(f)	60.00	–	–	–	0	3.70	66.95

8. Using the necessary headings from the chart in question 7, write a flow chart so that each of the following is the final printout. Part (a) is done for you.

(a) Profit
(b) Margin
(c) Cost price
(d) Overhead
(e) Loss
(f) Selling price



9·2 Slide Rule and Percent

Your work in this chapter will go much faster if you use your slide rule. If practice is necessary, work through the following exercise. If you experience difficulty ask your teacher for help.

EXERCISE 9·2

1. What percent is
   (a) 20 of 30? (b) 42 of 76?
   (c) 93 of 65? (d) 114 of 76?
   (e) 80 of 40? *Hint!* Read answer on left end of rule.

2. Find
   (a) 25% of 80 (b) 35% of 70
   (c) 65% of 150 (d) 16% of 96
   (e) 3% of 65 (f) 125% of 84
   (g) 145% of 114 (h) 133% of 75.

3. Find the missing number.
   (a) 21 is 30% of ■. (b) 26 is 65% of ■.
   (c) 312 is 48% of ■. (d) 60 is 30% of ■.
   (e) 73 is 43% of ■. (f) 85 is 72% of ■.

4. Find the missing number.
   (a) 34% of 74 = ■. (b) 26 is ■% of 96.
   (c) 68% of 24 = ■. (d) 84 is 44% of ■.
   (e) 76 is ■% of 44. (f) 86 is 125% of ■.

STRICTLY BUSINESS

"Our profit-sharing plan works both ways. I'm sending each of you a bill for what you owe the company."

9·3 Gross Profit

Most retail businesses allow for return of merchandise. The merchandise may be defective, it may be the wrong color or size, or the customer may just not want the item. Refunds for merchandise returned decrease the income from sales, therefore:

(6) Sales – refund for returns = net sales.

Once net sales are determined, the gross profit can be calculated by deducting the cost of the goods sold from the net sales.

(7) Net sales – cost of goods sold = gross profit

The cost of goods sold is not an easy item to determine. It is not usual to keep a record of the cost of each item sold. It is more usual to take inventory at the beginning and at the end of each recording period. The cost of the goods sold is then found by adding the cost of the purchases to the beginning inventory and subtracting from this sum the end inventory.

(8) Beginning inventory + New purchases – Ending inventory = Cost of goods sold

We relate the three terms net sales, gross profit and cost of goods sold in statement 7. Comparing statement 7 with statement 5 in Section 9 · 1, we note that gross profit is markup or margin.

Example 1

The following is an accounting for the month of July of the Big Chief Sports Shop.

Inventory as of June 30, 1972	$83,900
Purchases for July	12,300
Inventory as of July 31, 1972	81,200
Sales for July	19,600
Refund for returned goods	1,300

Find the gross profit for the month of July.

Solution

The Gross Profit Statement is presented here.

Big Chief Sports Shop
June 30, 1972 to July 31, 1972

Sales for July	$19,600	
Refund for returned goods	1,300	
Net sales for July		18,300
Inventory, June 30, 1972	83,900	
Purchases, June 30-July 31	12,300	
Saleable inventory	96,200	
Inventory, July 31, 1972	81,200	
Cost of goods sold		15,000
Gross profit		$ 3,300

EXERCISE 9·3

Calculate the gross profit by making a Gross Profit Statement in each situation.

1. The Teen Shop

Sales for September	$ 1,980
Refund for returned goods	196
Inventory, August 31	23,965
Purchases for September	2,116
Inventory, September 30	24,085

2. The Chocolate Shop

Inventory, November 30	$ 2,234
Purchases, December	894
Inventory, December 31	1,045.55
Sales for December	1,746.95
Refund for returned goods	3.25

3. The Fountain Bar

Sales for July	$991.78
Inventory, June 30	300.13
Purchases, July	751.42
Inventory, July 31	473.11

4. The Record Shop

Sales for the year 1972	$124,000
Refund for returned goods	3,500
Inventory at beginning of year	15,000
Purchases for year	94,500
Inventory at end of year	11,000

5. Give as many reasons as you can why goods might be returned for *money back*. Exchange lists with a classmate then do question 6.

6. If you were the manager of the store to which a customer returned goods for the reasons listed in question 5, which of the reasons would you find unacceptable? Explain to your classmate why you would not accept the reasons presented.

9·4 Markup

The owner of the Big Chief Sports Shop, Mr. Rod, in the previous section, knows that a markup (margin) of $1,300 is a suitable one. He wishes to determine what percent this is so that he can apply this rate to other sales. Can you help him? Would you calculate the percent markup on the basis of net sales or on the basis of cost of the goods?

Let us help Mr. Rod calculate the percent markup.

Based on net sales:

$$\text{Percent markup} = \frac{\text{markup}}{\text{net sales}} \times 100$$
$$= \frac{3300}{18300} \times 100$$
$$= 18\% \text{ (slide rule accuracy)}$$

Based on cost price:

$$\text{Percent markup} = \frac{\text{markup}}{\text{cost price}} \times 100$$
$$= \frac{3300}{15000} \times 100$$
$$= 22\% \text{ (slide rule)}$$

You will have noticed that it makes a difference on which base we calculate the percent markup. Now we are going to define carefully the difference between the two by using two terms.

rate of margin on sales
= percent increase based on net sales

rate of markup on cost
= percent increase based on cost.

NOTE! The terms *margin* and *markup* are not used interchangeably in these two cases.

Example 1

Mr. Rod buys 100 ski jackets at a cost price of $4,500. He knows he must have a rate of markup on cost of 20%. Assuming there are no returned goods,

(a) what is the sales price of the jackets?

(b) what is the rate of margin on sales?

Solution

(a) (9) $\text{Rate of markup on cost} = \frac{\text{markup}}{\text{cost price}} \times 100$

$$.20 = \frac{\text{markup}}{4500}$$
$$\text{markup} = \$900$$

Sales price is $4500 + $900 = $5400.

(b) (10) $\text{Rate of margin on sales} = \frac{\text{margin}}{\text{sales}} \times 100$

$$\text{Rate of margin on sales} = \tfrac{9}{54} \times 100$$
$$= 16.67 \text{ (slide rule)}$$

The rate of margin on sales is 16.67%.

Example 2

The cost price of a certain shipment of goods is $1,400. A merchant needs a 26% markup on cost. For what must he sell the shipment?

Solution

First method:

Since the markup is based on cost the total $1,400 represents 100%. Therefore, we must find 26% of $1,400 and add this sum to $1,400.

$$26\% \text{ of } 1400 = 364$$

$$\text{selling price} = \text{cost price} + \text{markup}$$

Therefore, selling price is $1400 + 364 = \$1{,}764$.

Alternate method:

$$\text{Since the cost price} = 100\% \text{ of cost}$$
$$\text{markup} = 26\% \text{ of cost}$$
$$\text{selling price} = \text{cost price} + \text{markup}$$
$$\therefore \text{selling price} = 126\% \text{ of cost}$$

Therefore, if we find 126% of 1400 we should have the selling price.

$$126\% \text{ of } 1400 = \$1{,}764 \qquad \text{(slide rule)}$$

Selling price is $1,764.

Example 3

The cost of a shipment of goods is $1,400. A merchant needs a 26% margin on sales. For what must the shipment sell?

Solution

Since the margin is 26% of sales

$$\text{sales} = 100\%$$
$$\text{margin} = 26\% \qquad \text{(cost price + margin = sales)}$$
$$\text{cost price} = 74\%$$

Therefore, $\dfrac{\text{cost price}}{\text{sales}} = \dfrac{74}{100}$

$$\frac{1400}{\text{sales}} = \frac{74}{100}$$

$$\text{sales} = \$1{,}890 \quad \text{(slide rule)}$$

The selling price is $1,890.

Equation Building by Guessing

The above solution is based on *formulas* that we have used in this chapter. In the event that we have forgotten the *formula* can we solve the problem? Let us use the method introduced in Book 2—*The Method of Equation Building by Guessing*. We shall solve the next example by this method.

Example 4

The cost price of a shipment is $60. A supplier decides he needs a rate of margin on sales of 25%.
(a) What is the selling price of the article?
(b) What is the amount of markup?
(c) What is the rate of markup on cost?

Solution

Let us guess what the selling price is: S. P. = $100. Now we must endeavour to verify our guess:

$$\begin{aligned} \text{selling price} &= \$100 \\ \text{cost price} &= \quad 60 \\ \text{margin} &= \$ \quad 40. \end{aligned}$$

We have been told in the question that the rate of margin on sales is 25%. Therefore,

$$\frac{\text{margin}}{\text{sales}} = \frac{25}{100}$$

(1) $$\frac{100 - 60}{100} \stackrel{?}{=} \frac{25}{100}$$

$$\tfrac{40}{100} \neq \tfrac{25}{100}.$$

Our guess is incorrect. Can we make a better guess? Test the guess. One soon realizes that guessing is an inefficient method. Therefore, let us use statement 1 and place a variable x in place of our guess each time it occurs in the statement; we get

$$\frac{x - 60}{x} = \frac{25}{100}.$$

Now solving for x

$$\begin{aligned} x - 60 &= 0.25x \\ x - 0.25x &= 60 \\ x &= \tfrac{60}{0.75} \\ &= 80 \end{aligned}$$

Selling price is $80.

Now that we have the selling price we can do the balance of the problem.

$$\begin{array}{ccccc} \text{selling price} & - & \text{cost price} & = & \text{markup} \\ 80 & - & 60 & = & 20 \end{array}$$

Markup is $20.

$$\begin{aligned} \text{rate of markup on cost} &= \frac{\text{markup}}{\text{cost}} \times 100 \\ &= \tfrac{20}{60} \times 100 \\ &= 33\tfrac{1}{3}\% \end{aligned}$$

The rate of markup on cost is $33\frac{1}{3}\%$.

EXERCISE 9·4

1. Find the rate of margin on sales and the rate of markup on cost in questions 1 through 4 of Exercise 9·3.

2. If a merchant buys $940 worth of goods and wants a 30% markup on cost, what must he receive in sales for this shipment?

3. If the merchant in question 2 wants a 30% margin on sales what must he receive in sales for this shipment of goods?

4. From the information in each case, calculate:
   (a) the margin (markup)
   (b) the rate of margin on sales
   (c) the rate of markup on cost.

	(i)	(ii)	(iii)	(iv)	(v)
selling price $	75	132.50	765.00	497	1445
cost price $	65	106.30	603.80	387	1023

5. A motorbike dealer receives a shipment of bikes. He operates on a markup on cost of 24%. If one bike is listed at $320, determine:
   (a) the selling price
   (b) the markup
   (c) the rate of margin on sales.

6. A supplier receives a shipment of dresses. The dresses are listed at $34.50 after being marked up 15% of cost. Calculate:
   (a) the markup
   (b) the cost price
   (c) the rate of margin on sales.

   Hint! Since this rate is based on cost, the cost price is 100%. (See Example 2.)

7. When the selling price of a new record album was $6.50 the dealer's profit was 36% of the selling price.
   (a) What is the cost price of the record album?
   (b) What is the margin?
   (c) What is the rate of markup on cost?

8. Complete the table.

COST PRICE	MARKUP OR MARGIN	SELLING PRICE	RATE OF MARGIN ON SALES	RATE OF MARKUP ON COST
$ 2.16	$.	$.	25%	.
.	.	40.50	30%	.
36.40	.	.	.	35%
.	.	72.40	.	25%
.	6.00	.	22%	.
.	8.50	.	.	34%

9. Why might a businessman prefer to calculate *markup* on sales rather than cost?

10. Why might a businessman prefer to calculate *markup* on cost rather than sales?

9·5 The Profit and Loss Statement

A businessman must constantly be assessing his business. He must be aware of the many aspects of his business: the sales, the overhead, the cost of goods sold, and his profits. He must assess his complete business operation periodically. In order to obtain an overall financial picture of his business, he summarizes all the information in a *Profit and Loss Statement.*

The general form of the Profit and Loss Statement is shown in the example that follows.

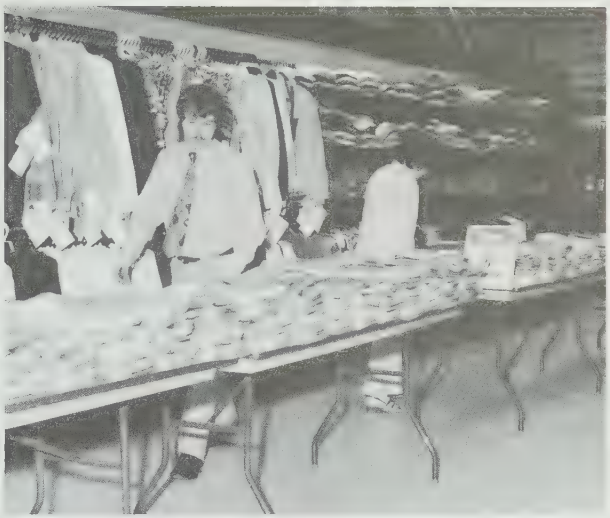
Mr. Suits' Clothing Store
Edgewater, Ontario

Statement of Profit and Loss
for the month ending June 30, 1972

Gross sales	$52,800
Less: returned goods	2,200
Net sales	50,600
Less: cost of goods sold	39,200
Gross margin	11,400
Less: Overhead	9,900
Net profit	1,500

The above statement is derived from records kept in the accounting department. We shall not be concerned with the separate accounts that produce net sales, cost of goods sold, and overhead. The importance of the Profit and Loss Statement is in its analysis. The owner or manager will study the relation between the various items of the statement and these relations will be compared with those from previous years or with those from firms in similar situations.

Let us prepare an appropriate analysis sheet. We shall place a percentage figure beside each of the amounts in the Profit and Loss Statement.



It is most useful to the businessman to express each part of the Profit and Loss Statement in terms of its percent of the *net sales.* It is these ratios (percentages) that are compared with corresponding ratios from other firms or other years. Certain types of firms set up standard ratios which they attempt to achieve each year. If the actual ratios fall below, or rise too high above the *standard ratios,* attempts must be made to draw them back into proper proportions. If competitive pricing is not a factor (it usually is), the selling price is one factor that can be adjusted to bring ratios closer to the standards.

The analysis of the Profit and Loss Statement for Mr. Suits' Clothing Store is shown here:

Analysis

(a) Net sales – base – 100%

(b) Cost: $\frac{39,200}{50,600} \times 100 = 77.5\%$

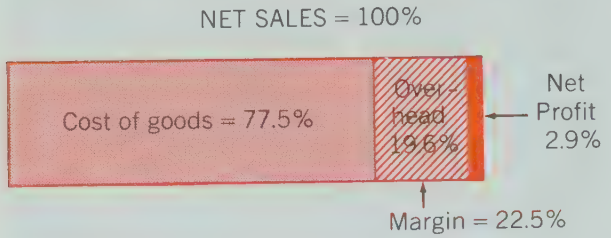
(c) Margin: $\frac{11,400}{50,600} \times 100 = 22.5\%$

(d) Overhead: $\frac{9,900}{50,600} \times 100 = 19.6\%$

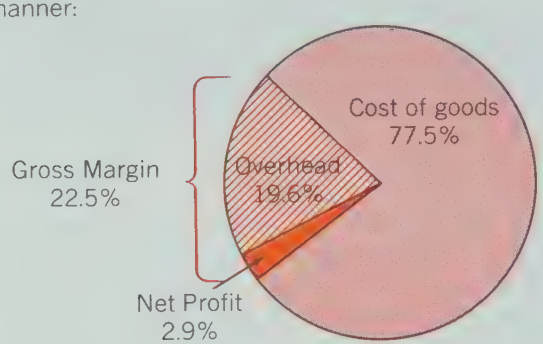
(e) Net profit: $\frac{1,500}{50,600} \times 100 = 2.9\%$

Note: (b) + (c) = 100
(d) + (e) = (c)

A rectangular graph shows the data obtained from the analysis:



A circle graph shows the same information in another manner:



To draw the rectangular graph above:

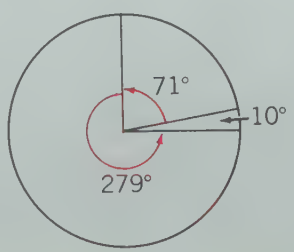
	Actual diagram
(a) draw a rectangle x units long (any width);	7 cm
(b) find 77.5% of x units– measure off this rectangle;	5.4 cm
(c) find 19.6% of x units– measure off this rectangle;	1.4 cm
(d) find 2.9% of x units– measure off this rectangle.	0.2 cm

rounded off

To draw the circle graph:

The full circle, 360°, represents 100%

(a) draw a circle and one radius;
(b) 2.9% of 360° = 10°.
Measure of 10°.
Draw second radius.
(c) 19.6% of 360° = 71°
Measure of 71°
from second radius.
Draw third radius.
(d) 77.5% of 360° = 279°.
Check.



EXERCISE 9·5

1. Complete this Profit and Loss Statement.

BIG WHEELS
THE COMPLETE Motor Bike Shop

Statement of Profit and Loss
for the month ending June 30, 1972

Gross sales	$26,900
Less: returns	nil
Net sales	?
Less: cost of goods sold	23,200
Gross margin	?
Less: Overhead	3,350
Net profit	?

2. Prepare an analysis of the Profit and Loss Statement in question 1.

3. Prepare a rectangular graph to represent the analysis from question 2.

4. Prepare a circle graph to represent the analysis from question 2.

5. Why do you think the data from the Analysis Statement is presented in a rectangular graph or a circle graph?

6. Make a Profit and Loss Statement and analyse it as illustrated in the examples above for each of the following sets of information.

	Gross sales	Returned goods	Cost of goods sold	Overhead
(a)	$44,800	$1,400	$ 29,700	$ 8,900
(b)	98,700	3,300	57,000	29,300
(c)	146,100	2,100	100,000	34,000
(d)	77,700	900	42,200	10,900

7. Answer the following questions as though you were the manager or owner of a business.

   (a) What figures in the Profit and Loss Statement are of primary interest?

   (b) If the ratio of the net profit to the net sales started to fall below the standard ratio or started to fall in general, what are some of the things you could do to keep the ratio steady or to increase the ratio?

   (c) If the ratio of the gross margin to the sales started to rise substantially above the standard ratio, would you be fearfully alarmed or pleasantly concerned? Why? What would you do – let it continue to rise, endeavour to lower it, or endeavour to make it rise higher still? What method might you use to achieve your objective?

   (d) If the net sales figure rose sharply but the ratio of net profit to net sales dropped slightly, would there necessarily be cause for alarm? Explain.

8. The following information was taken from a firm's accounting office at 6 month intervals. Set up a Profit and Loss Statement with appropriate analysis for each set of information. If you were manager or owner what would be your reaction to the trend?

Date	Gross sales	Returned goods	Cost of goods sold	Overhead
(a) Jan. 1	$57,400	$1,100	$38,800	$6,900
(b) July 1	80,000	9,000	49,500	8,700
(c) Dec. 31	89,400	1,800	62,400	9,900

9. From the information given in question 8 answer the following.

   (a) Using January 1 data as the basis, what percent increase was there in gross sales as of July 1? as of December 31?

   (b) Using January 1 data as the basis, what percent increase was there in net sales as of July 1? as of December 31?

   (c) Would you, as owner, be alarmed at the answers to questions (a) and (b)? Why?

10. Obtain from the manager of the school canteen the inventory statements, the purchases, the sales, and refunds for returned goods, for a given period of time. Also, obtain the overhead.

    (a) Prepare a Statement of Profit and Loss for the canteen. Prepare an analysis of the statement.

    (b) Repeat this question monthly for several months and prepare a recommendation for the canteen manager.

11. Ask some members of Junior Achievement to explain to the class how this organization operates. If interest is sufficient some officials of this organization may be asked to speak to a class or school assembly.

12. Do you think this dealer is losing money on these cars?

9·6 The Retail Business

The retailer sells commodities directly to the consumer. Wholesalers and manufacturers (suppliers) deal with retailers or with other wholesalers or manufacturers. Retail businesses can be grouped under two headings: those that buy and sell goods only, such as hardware stores, grocery stores, clothing stores, and those that deal in goods and provide services also, such as garages, restaurants, electrical sales and service shops.

Retailing consists of buying at one price and selling at a higher price. Usually it consists of buying in large quantities and selling in smaller quantities.

A school canteen is a good example of a retail business. It differs from most businesses in two main ways:

1. It operates primarily as a service to the student body and the profit motive is secondary. In the business world the profit motive is primary. Service to the public is provided as a means to increase business.
2. Labour is often free or at a very minimal cost in the canteen situation. In business labour is a very large cost item.

The manager of the canteen receives from the suppliers (wholesalers or manufacturers) a catalogue of their products. In the catalogue all items are shown with a *suggested list price* or *catalogue price*. Accompanying the catalogue is a *trade discount sheet*. The canteen pays the *net price*, or *discount price* for the goods.

Net price = list price − discount

Example 1

Chocolate bars are listed at 10 cents each. The trade discount is 30%. Find the net price of 400 bars.

Solution

List price	$40	(400 ×.10)
Discount	$12	(30% of 40)
Net price	$28	

Note: Since the discount is 30% the net price must be 70% of the list price.

70% of $40 = $28

Why is there a list price and a trade discount? Suppliers must change their prices often since costs fluctuate monthly, weekly or sometimes daily. If the supplier listed his true price then the catalogue would have to be changed with every change in this price. But since the suggested list price in the catalogue is accompanied by a discount sheet the supplier needs only to print and distribute a new discount sheet.

In some cases the suggested list price is the advertised retail price. The supplier allows a discount so that the retailer can realize a satisfactory gross profit. The retailer's gross profit on the item is the trade discount provided he sells it at the suggested list price.

Successive Discounts

It is not uncommon for a supplier to offer other discounts: discounts on large quantities, discounts for cash payments, discounts to promote sales or to introduce new lines, and so on. Usually each additional discount is a *successive discount.* That is, the first discount is calculated, the reduced price is calculated, and then the *second discount* is calculated *on the basis of the reduced price.*

Example 2

Whitfield and Company, suppliers, list the new nylon tip pens for 80 cents each. The trade discount is 30% with a 10% promotion discount. What is the net price to the canteen of 100 pens?

Solution

The list price is $\$.80 \times 100 = \$80$.
The first discount is based on the list price of $80.

List price	$80.00	
Trade discount	$24.00	(30% of 80)
Reduced price	$56.00	
Promotion discount	$ 5.60	(10% of 56)
Net price	$50.40	

The net price is $50.40

EXERCISE 9·6

Find the net price in each case.

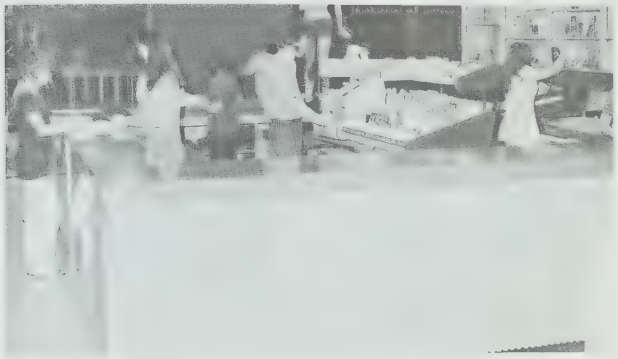
	List price	*% Trade discount*	*% Second discount*
1.	$ 39.40	35	
2.	144.50	28	
3.	4.20	40	
4.	48.55	32	
5.	24.00	25	10
6.	74.00	35	15
7.	450.00	26	14
8.	390.00	45	8
9.	564.00	40	12
10.	1,440.00	42	6

11. The Sir Winston Churchill High School canteen ordered

    (a) 400 chocolate bars listed at $0.10 with a 20% trade discount and a second discount of 4%.

    (b) 300 graph books listed at $0.25 with a trade discount of 40% and a second discount of 5%.

    Find the total net price the canteen would pay the supplier.



12. Why would a supplier offer successive discounts? Name as many reasons as you can.

13. Ask the manager of your school canteen to give a report to your class of a month's operation. Ask him to show you the statements from the suppliers.

14. The Queen Elizabeth High School canteen's monthly statement contained the following items:

    Soft drinks

List price	$250
Net price	$160

    What was the percentage trade discount?

Find the percentage trade discount in each instance.

	List price	*Net price*
15.	$ 400	$ 240
16.	$2800	$1960
17.	$2700	$1500

18. Represent graphically the data in Example 2 in as many ways as you can. *Hint!* Can you show the various prices? percentages? discounts?

9·7 Effects of Successive Discounts

In the previous section we calculated successive discounts. In this section we will calculate the single discount that is equivalent to a series of discounts. Then we will use this equivalent discount to solve certain problems.

Example 1

Find the single discount equivalent to the successive discounts of 40%, 20% and 5%.

Solution

The list price	=	100% of itself
Less 40% of 100	=	40% of list price
1st reduced price	=	60% of list price
Less 20% of 60	=	12% of list price
2nd reduced price	=	48% of list price
Less 5% of 48	=	2.4% of list price
Net price	=	45.6% of list price

A single discount of 100% – 45.6% = 54.4% is equivalent to the discount series.

Example 2

The net price of an article after a trade discount of 35% and a second discount of 6% was $144. What was the list price?

Solution

List price	=	100% of itself
Less 35%	=	35% of list price
Reduced price	=	65% of list price
Less 6% of 65	=	3.9% of list price
Net price	=	61.1% of list price

61.1% of list price is $144.

100% of list price is x.

Using slide rule, the list price is $236.00.
(Actual calculation: $235.68)

List as many reasons as you can why people buy at sales.

EXERCISE 9·7

Compute the single discount equivalent to each of the series of discounts.

1. 35%, 10%
2. 25%, 20%
3. 50%, 50%
4. 30%, 20%, 10%
5. 45%, 10%
6. 10%, 10%

Calculate the list price in each of the following.

	Discount series	*Net price*
7.	25%, 10%	$ 250
8.	30%, 20%	$ 500
9.	40%, 40%	$ 400
10.	10%, 10%, 10%	$1,000

11. A retailer may buy from one of three wholesalers – A, B, or C. Each wholesaler offers a different discount series:

    A. 40%, 20%, 10%

    B. 10%, 20%, 40%

    C. 10%, 40%, 20%

    If you were the dealer, from which wholesaler would you buy? Why? Support your reasons by calculating the single discount equivalent to each series.

12. Why would a merchant want to discount or mark down goods in a store? Consider Mr. Suits in his clothing store; a car dealer; a sportsgoods dealer; a grocery store manager.

13. What is your comment on this Pre-Christmas $\frac{1}{2}$ Price Special?

Pre-Christmas ½ Price Specials

INTRODUCTORY OFFER

TONERS $6.50 beautiful fall shades

Shampoo and Set $2.50

No extra charges

New Protenized Cold Wave with Bonedium, gives extra body. Reg. 25.00 ——— $12.50

Hair manager so natural. Reg. 15.00 ——— $10.00

Zotos very special. Reg. 12.50 ——— $7.50

BEAUTY SALON

14. This ad appeared in a local newspaper.

CLOSING OUT SALE

Each Monday after January 3, there will be a further reduction of 10% of the remaining price of each car unsold.

1965 DODGE SEDAN
V-8, auto. trans., outstanding one owner unit.
$1175

1968 DODGE POLARA
500 HARDTOP — full power equipped, radio, automatic. As new.
$2750

1967 PLYMOUTH FURY III
CONVERTIBLE — full power, equipped, radio, auto. trans., factory warranty.
$2210

DON'T WAIT TOO LONG.
VISIT TODAY
H & W MOTORS

Prepare a list showing the prices of the three cars on the three Mondays following January 3.

15. Which car shows the largest percentage markdown?

DEAL MAKER

Call now

REDUCED FOR THIS SALE

66 **Chevrolet Sedan,** V-8, Automatic, Radio, Reconditioned, O.K. Warranty. C1176A. Was 1495.00 Now ——— $1295

66 **Chevrolet Impala 4 door Hardtop,** V-8, Automatic, Radio, Power Steering, Power Brakes, Reconditioned, O.K. Warranty. E1688A. Was 1895.00 Now ——— $1695

67 **Chevrolet Bel Aire Sedan,** V-8, Automatic, Radio, Power Steering, Power Brakes, Reconditioned. O.K. Warranty. FL1991A. Was 2095.00 Now ——— $1895

63 **Chevrolet Impala Sedan,** V-8, Automatic, Radio, Power Steering, Power Brakes, Reconditioned. O.K. Warranty. F1006A. Was 995.00 Now ——— $795

68 **Chevrolet Sedan,** Reconditioned, New Car Warranty. FL2066A. Was 1895.00 Now ——— $1695

9·8 Cash Discounts

The retailer orders his stock by telephone, ordinary letter, order form provided by the supplier, or the retailer's own order form. The goods are subsequently shipped and received. The supplier sends an invoice, usually through the mail. Invoice forms vary from dealer to dealer but all contain a list of the goods shipped and the respective costs.

The retailer checks the items received against the items listed on the invoice. He checks the amounts charged. Finally, he approves the payment and indicates how it is to be paid. If the retailer operates a small business he may make the payments himself. Otherwise, a designated empioyee attends to the matter.

The following is a typical invoice statement.

CANTEEN SUPPLIES LIMITED
WINNIPEG MANITOBA

Sold to: Sir Winston Churchill
High School
CANTEEN
3516 Ford Ridge
WINNIPEG, Manitoba

1234 Customer No.	10 Mo.	19 Day	72 Yr.	70-3561 Invoice No.

Ship to: Same

TERMS 2/10, n/30

1% per month payable on overdue account.

Via: Truck

Shipping Instructions

Quantity Shipped	Quantity Ordered	DESCRIPTION	UNIT PRICE	NET AMOUNT
100 dz.	100 dz.	Chewy mints	$1.20 per dz.	$ 120.00
50 dz.	50 dz.	Pure chocolates	1.20 per dz.	60.00
30 gr.	30 gr.	Pencils--HB-- eraser	7.00 per gr.	210.00
				$ 390.00

Figure 9-1

The invoice shows the terms of payment: 2/10, n/30. 1% per month payable on overdue amounts.

2/10 means 2% discount on any payment made within 10 days of the date of the invoice.

n/30 means the customer is to pay the balance owing on the invoice within 30 days.

1% per month payable on overdue accounts refers to the service charge on any balance unpaid after 30 days.

There are other terms used also:

Net 30 days means there is no cash discount and the payment is due in 30 days.

2% 15th prox. means a 2% discount on payment made by the 15th of the month following the one in which the sale was made. Prox. is the abbreviation for *proxima* which means the month following the present month.

n/e.o.m. or n/EOM means net due no later than the *e*nd *o*f the *m*onth after that in which the sale was made.

The due date of an invoice is found by counting ahead from the date of the invoice. If the invoice shows n/60 and the date of the invoice is January 21, then the payment is due 60 days later, on March 21.

60 days
−11 days left in January
49
−28 days left in February
21 days in March

Amount payable = invoice total − cash discounts

Cash discount = invoice total × rate of discount

Example 1

Find the amount payable if the invoice from CANTEEN SUPPLIES LIMITED shown in Figure 9-1, is paid on

(a) February 10 (b) March 5 (c) March 25.

Solution

(a) Since February 10 comes within the 10-day discount period, the invoice is discounted 2%.

Therefore,

$$\begin{aligned}\text{amount payable} &= 390 - (390 \times .02)\\ &= 390 - 7.80\\ &= \$382.20.\end{aligned}$$

(b) March 5 does not fall within the 10-day discount period, but does fall within the 30-day period. Thus the amount payable is the full amount of $390.

(c) March 25 is beyond the 30-day period and is within the first month over the 30-day period, therefore 1% of the total is added to the total.

$$\begin{aligned}\text{Amount payable} &= 390 + (390 \times .01)\\ &= 390 + 3.90\\ &= \$393.90\end{aligned}$$

The trend today is away from cash discounts. Instead, many firms simply state net due date, i.e., n/30 or n/prox., with a penalty clause such as 1% per month on unpaid balance. In a penalty clause, if the payment is 1 day or more overdue then the penalty is assessed for the whole month. For instance, if the payment was due May 20 but it was made on May 21, then the penalty is charged for a whole month. Consequently, the retailer may wait until June 19 before making payment and not be assessed any more than he was on May 21.

EXERCISE 9·8

Calculate the cash discount, penalty, if any, and net amount paid in each case.

	Invoice total	Date of invoice	Terms	Date paid
1.	$380.00	Jan. 4	2/10, n/30	Jan. 12
2.	452.00	Jan. 20	5/10, n/30	Feb. 25
3.	35.50	Feb. 10	5/10, 2/30, n/60	Mar. 1
4.	450.60	Mar. 15	2% 10th prox. n/30 prox.	Apr. 2
5.	762.40	Mar. 20	5%/15 prox. n/EOM	Apr. 18
6.	390.50	May 2	n/30 – 1% per month on balance	June 30
7.	897.65	July 4	5/15, n/30	July 16

8. A retailer received an invoice dated August 3. The total was $1,490 and the terms were 5/10, n/30 with a 1% per month on unpaid balance. The retailer made the following payments: August 11 – $600; August 30 – $600; September 30 – balance. Find the balance that the retailer had to pay in order to clear the account.

9. Why do you think there is a trend away from cash discounts?

Projects

Project 1

Arrange with the manager of the school canteen to provide the class with his invoices, statements, and inventories for a month. Duplicate copies of each for the class. Each class member then prepares the appropriate month end reports and statement for the canteen manager.

In each group decide what statements you will need at the beginning of the month, at the end of the month, and what statements you will prepare.

Project 2

Loss Leaders are those items that a manufacturer or retailer sells at a loss in order to promote sales on some other items. A camera and film manufacturer may put a simple camera on the market at a loss in order to sell film.

A retailer may use *limited quantities sales, one per customer, door opening specials* or *hour specials*, to get customers to his store. The retailer hopes that while the customer is in the store, he will do some *impulse buying*.

at JT's Hardware
SATURDAY SPECIALS
WHILE STOCKS LAST
9 A.M.
Bapco interior latex paint. 48 gallons offered. Sale, ea. 1.96
4-ft. wooden step ladder. 40 offered. Sale, ea. 3.33
Mattress or box spring. 20 offered. Sale, ea. 9.66
10:30 A.M.
Hardwood briquets. 10-lb. bag. 50 offered. Limit: 2 per customer. Sale, ea. 39¢
10:30 A.M.
Fondue sets. 20 offered. Sale, ea. 4.99
11:30 A.M.
Guitars. 24 offered. Sale, 9.66
1:30 P.M.
Fishing lures. Sale, 9 for 99¢
Spinning reels. 50 offered. Sale, 1.76
Solid spin cast rods. 50 offered. Sale, 4.96

Arrange with your teacher to visit the sales manager of a large department store. Ask the manager to explain the reason his store has various types of sales. Ask him which is the most profitable.

Compile a number of questions you will want to ask the sales manager. Also ask the manager which items in his store are loss leaders. If you are unable to explain why each is a loss leader ask him to explain.

9·9 Promissory Notes

In operating a business, it is frequently necessary to borrow money to meet certain costs. If these costs are not paid the debtor may suffer

1. by obtaining a *poor-credit-risk* reputation.
2. excessive interest charge.
3. court action against him.

The manager of the business may arrange to give a promissory note to the person to whom he owes the money.

A *promissory note* is a statement in which one person or organization promises to pay another person or organization a certain sum of money. The person or organization to whom the money is owed is called the *creditor*. The person or organization that owes the money is called the *debtor*.

Study the following note.

The *date of maturity*—the date the note is due—is June 10, 1972. The *date of the note*, May 8, 1972, is the date on which the note was written. The *time* is thirty days. Notice that the time is 30 days, yet from May 8 to June 10 is 33 days. In Canada the date of maturity is found by taking into account the time of the note plus *three* additional days called *days of grace*.

John Winston and Company to whom the note is payable, is called the *payee*. Simon Richards, who signed the note is called the *maker*.

The amount to be paid is called the *face of the note*. It is common to write this twice, once in numerals and once in words.

Do question 1 in Exercise 9·9.

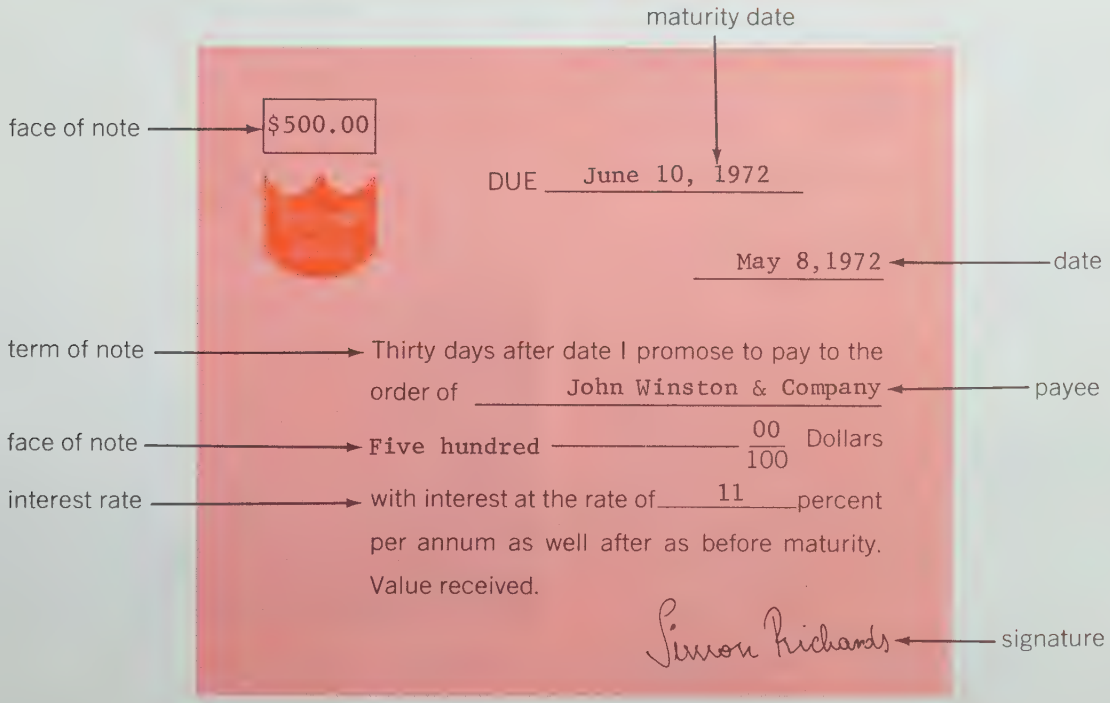


Figure 9-2

To Find the Date of Maturity

The time of a note may be stated in days, months, or years. To calculate the date of maturity, study the three examples.

Example 1

Find the date of maturity of a note dated August 13, 1970 if the time of the note is 90 days.

Solution

Time = 90 days

Days in August	18 days left 72
Days in September	30 days left 42
Days in October	31 days left 11
Days in November	11 days left 0

November 11 is the 90th day but since 3 days of grace are to be included, the maturity day is November 14, 1970.

Example 2

Find the date of maturity of a 3-month note dated March 22, 1971.

Solution

Count 3 months forward from March 22:

March 22 to April 22	(1 month)
to May 22	(2 months)
to June 22	(3 months)

Now add 3 days of grace and the date of maturity is June 25, 1971.

Note! If the note were dated November 31, 1970 the note would be due March 3 since there is no February 31. The rule in this case is:

> If the month of maturity does not have such a date, the date of maturity is the last day in the month of maturity, plus three days of grace.

Hence, November 31 to February 28 is 3 months.

When we add the 3 days of grace the date of maturity becomes March 3, 1971.

The note illustrated in Figure 9-2 is an interest-bearing note which means the face of the note states the amount owed and includes a statement which tells the rate at which the interest is to be calculated. In the illustration (Figure 9-2) the interest is 11% per annum until the date of maturity. If the note is not *honoured* (paid), then interest is charged at the rate of 11% per annum until the note is honoured.

To find the interest on an interest-bearing note, the exact number of days is counted and used as the time.

Example 3

Find the amount payable on a $450, 3-month note dated January 8, carrying an interest charge of 10% per annum.

Solution

Find the due date: Due date is April 11.
Note! Interest is charged on 3 days of grace.

Find the exact number of days:

days in January	23
days in February	28
days in March	31
days in April	11
Total number of days	93

Find the interest:

$$I = prt$$
$$= 450 \times .10 \times \frac{93}{365}$$
$$= \$11.47$$

Amount payable is $450 + $11.47 = $461.47

Do questions 2 to 7 in Exercise 9·9.

Discounting Notes

If the payee of a promissory note is in need of money before the date of maturity, he may sell the note to his bank (or some other financial agency) and receive cash for it. The bank is *advancing* the payee money on the note. The bank will require rent (interest) on the advance. This interest is called *bank discount.* The date the payee sells the note to the bank is called the *discount date.* The bank will hold the note until the maturity date, then collect the maturity value. The number of days from the discount date to the maturity date is called the *discount period.* Selling a note before its maturity date is called *discounting the note.*

John Winston and Company presented the note in Figure 9-2 to a bank on May 15, 1972 to be *discounted.* The bank does not pay John Winston and Company the full maturity value. The bank will not receive its money until June 10. Therefore, it is actually lending money to John Winston and Company. The interest is collected in advance.

Let us assume the bank discounts the note at 8% per annum. This means, it will pay John Winston and Company the maturity value less 8% for the balance of the time of the note (for the discount period). The maturity value of this note is \$504.97. The bank discount is calculated as follows:

Number of days from May 15 to May 31 (incl.) is 16
May 31 to June 10 (incl.) is 10
Total days 26

We could use

$$\text{Interest} = \text{principal} \times \text{rate} \times \text{time}$$

but we use different terms for discounting. Actually, the new formula in effect is just the interest formula.

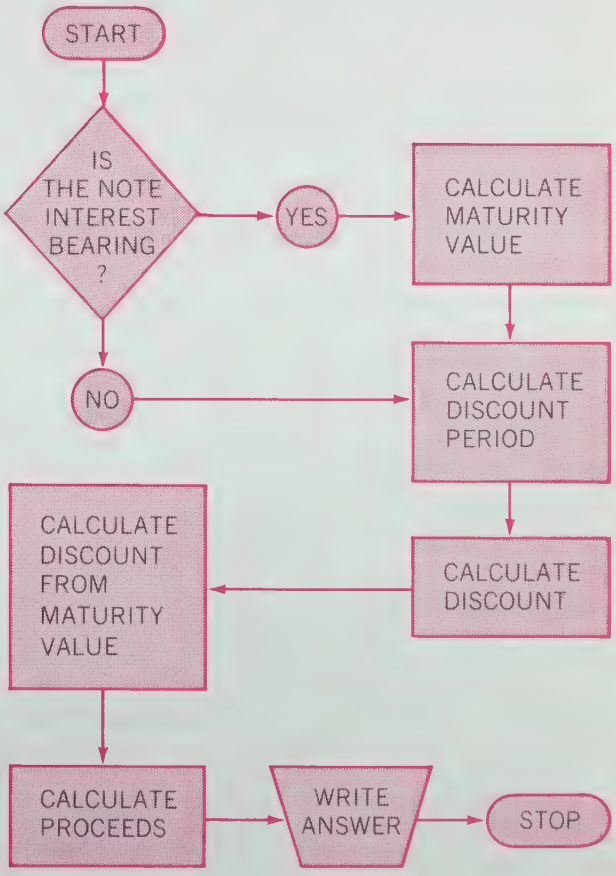
(1) Discount = maturity value × discount rate × discount period

$$\text{Discount} = 504.97 \times .08 \times \tfrac{26}{365}$$
$$= \$2.88$$

$$\text{Proceeds} = 504.97 - 2.88$$
$$= \$502.09$$

The amount (called proceeds) John Winston and Company receives is \$502.09.

A flow chart showing how to calculate the proceeds of any discounted note is shown.



EXERCISE 9·9

1. Refer to this note to answer the questions.

$300.00 DUE September 23, 1970

November 20, 1969

Ten months after date I promise to pay to the order of

J. R. T. Togans Ltd.

----Three hundred--------- 00/100 Dollars

with interest at the rate of 9½ percent per annum as well after as before maturity.

Value received. R. T. Morris

(a) Who is the payee? the maker?
(b) What is the face of the note?
(c) What is the date of the note?
(d) What is the date of maturity?
(e) What is the interest rate?
(f) What is the time of the note?

2. Find the date of note or the date of maturity.

	Date of note	*Time*	*Date of maturity*
(a)	3 Jan., 1970	45 days	?
(b)	20 Nov., 1971	6 months	?
(c)	17 March, 1970	3 Years	?
(d)	?	120 days	18 Sept., 1970
(e)	?	6 months	31 Aug., 1969
(f)	10 July, 1969	10 months	?
(g)	30 Aug., 1970	6 months	?

3. Compose a promissory note for the following situation.

G.P. Montague owes J. Oulette $640. A 60-day note is dated August 10, 1970. The note carries a $7\frac{3}{4}\%$ interest charge both before and after maturity.

4. Find the amount payable on the date of maturity of each note.

	Face of note	*Date*	*Time*	*Rate*
(a)	$325	April 10	3 months	8%
(b)	$750	May 15	70 days	$8\frac{3}{4}\%$
(c)	$1000	January 14	2 years	$11\frac{1}{2}\%$
(d)	$955	November 9	1 year	$9\frac{5}{8}\%$
(e)	$644	June 4	165 days	7.3%

5. In order to pay the taxes on his home and to avoid the penalty for late payment, Mr. Marrs borrows $658 on February 12 and signs a 4-month note bearing interest at the rate of $8\frac{1}{2}\%$. What is the total he must pay on the date of maturity?

6. Find the maturity value of the note illustrated in Figure 9-2.

7. Find the maturity value of the note in question 1.

8. Find the proceeds on each note in question 4 if the respective notes are discounted on the dates shown. (Let discount rate = 8%.)

   (a) April 16 (b) May 31

   (c) January 31 (d) January 25

   (e) June 14 (f) March 31

9. Find the proceeds of the notes shown below.

	Face of note	Date of note	Interest rate	Time of note	Date of discount	Discount rate
(a)	$300	Aug. 1	7%	60 days	Aug. 15	8%
(b)	$856	Mar. 14	11%	2 months	Mar. 31	10%
(c)	$436	Aug. 16	nil	90 days	Aug. 31	$9\frac{1}{2}$%
(d)	$1700	July 4	nil	45 days	Aug. 30	$8\frac{3}{4}$%

10. Extend an invitation to a bank official to talk with your class about promissory notes. Prepare a list of questions and topics your class will ask to be discussed.

Using The Library

1. *The Giant Golden Book of Mathematics* Read pages 84-85. How was Archimedes killed? Archimedes proposed he could move the earth. How did he propose that he could do this?

   Read pages 86-87. Newton once said "If I have seen farther than others it is because I have stood on the shoulders of giants." Explain what he meant by this.

2. *The Mathmen* Read pages 85-90. What are the Platonic Solids? Why did Plato conclude that the number 5 rules the universe?

   Read *Mathmen*, pages 163-167 and 181-184. To what did Archimedes refer when he cried "Eureka"? What was Archimedes' contribution to the Second Punic War?

3. *The Giant Golden Book of Mathematics* Read page 82. Try this card trick with a deck of cards. Can you make it work?

REVIEW EXERCISE 9

1. Calculate the gross profit by making a Gross Profit Statement.

   The Blue Notes Musical Instrument Shop

Sales for January	$183,670
Refund for returned goods	2,670
Inventory, December 31	22,300
Purchases, January	140,100
Inventory, January 31	23,900

2. Goods with cost price of $120 are subject to a rate of margin on sales of 35%.

   (a) What is the selling price of the goods?

   (b) What is the margin?

   (c) What is the rate of markup on cost?

3. A jeweller offers a $39.80 watch for $27.50. What is the rate of markdown?

4. The Wig Shop
Centerville County

Statement of Profit and Loss
for the month ending November 30, 1972

Gross sales	$32,600
Less: returns	1,100
Net sales	
Less: cost of goods sold	16,250
Gross margin	
Less: Overhead	12,500
Net profit	

(a) Complete the Profit and Loss Statement.

(b) Prepare an analysis of the Profit and Loss Statement.

(c) Prepare rectangular and circle graphs to represent the analysis from part (b).

(d) If you were the proprietor (or salesman) of The Wig Shop would you be satisfied with the business? Why?

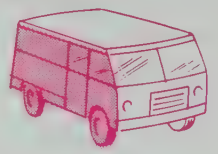
5. Company A offers a trade discount of 40% followed by a discount of 22% on a basketball uniform listed at $40. Company B offers a single discount of 51%. Which offer is the better?

6. Find the date of a note when the date of maturity is May 31, 1972 and the time is 3 months.

7. Find the amount payable on the date of maturity of a note dated March 23 if the face of the note is $730, the rate is $6\frac{1}{2}\%$ and the time is 4 months.

8. Find the proceeds of a note for $960 dated December 16; interest rate is nil; time is 45 days; date of discount is December 31 and discount rate is 5%.

SPOTLESS CLEANERS



Our company provides many different services. In the enclosed example we will provide you with approximate figures pertaining to our drycleaning and laundry service. However, you might wish to include other examples such as: shoe repairs, suede and leather cleaning, paid repairs, and feather pillow cleaning.

Below you will find a simplified version of our sales analysis sheet. Basically this sheet tells us how our weekly sales volume compares to the same week the previous year.

WEEK NO. 1 ENDING JANUARY 6/68

Route Number	Dry Cleaning 1968	Dry Cleaning 1967	Laundry 1968	Laundry 1967
R1	400.10	385.20	200.20	180.10
R2	350.40	360.10	150.60	158.60
R3	375.60	405.80	250.25	140.10
R4	410.80	320.10	310.40	300.50
R5	495.20	405.06	400.50	405.50
R6	308.09	300.80	185.95	215.00
R7	200.05	200.20	165.40	184.20
R8	325.10	310.40	200.40	225.60
R9	175.00	125.00	104.08	150.40
R10	240.00	200.00	95.40	98.10

	Total Volume 1968	Total Volume 1967	% Increase or Decrease 1968 to 1967	
R1	600.30			
R2	501.00			
R3	625.85			
R4	721.20			
R5	895.70			
R6	494.04			
R7	365.45			
R8	525.50			
R9	279.08			
R10	335.40			

Chapter Ten
MUNICIPAL TAXATION

10·1 Municipal Taxation

No doubt where you live you have roads, police protection, educational facilities, health services, and perhaps, sidewalks, water and sewage facilities, and certain social services. Who pays for these and the many other services you may have available? If you live in a village, town, or city, the local government is the village, town, or city council. If you live in a rural area you may be under a county or municipality. Some rural areas of the county are directly under provincial or federal jurisdiction.

Generally, we refer to all local government as *municipal government.* This is in contrast to *provincial* or *federal government.* We then refer to municipal taxation as *local taxation* administered by local government.

Each year the council of the municipality prepares a budget of expenditures and revenues for the next year. The major portion of the revenue money must come from the people who will enjoy the services provided. This money is raised by charging or assessing property owners. This charge is a *tax.* Another source of money is the *business tax.* The business tax is paid in addition to the *property* or *realty tax.* Business tax will be discussed later in this chapter. Minor sources of money are fines, licences, permits and fees.

Real Property Tax

Real property tax, usually called *property tax,* is levied on real property. By *real property* is meant land and buildings.

For taxation purposes, each piece of real property is assigned a value. This assigned value is usually some set percentage of a fair value of the property at the time of the inspection. An *assessor,* specially trained and employed by the municipality, does the assigning called *assessing.* The value he assigns is called the *assessment.* Since property value changes from time to time each municipality revises its assessments periodically. Once the assessments are made and the amount of money needed from this source is determined, the council calculates the tax rate.

(1) $\text{Taxes} = \text{assessment} \times \text{rate}$

Thus, if a municipality has a real property assessment, of $350,000,000 and it budgets for $21,350,000 from real property tax then the rate is calculated as shown:

$$\text{rate} = \frac{\text{taxes}}{\text{assessment}} \qquad \text{(from formula (1))}$$

$$= \frac{21{,}350{,}000}{350{,}000{,}000}$$

$$\text{rate} = \$0.061 \text{ per dollar.}$$

We can say the tax is $0.061 on each dollar of real property assessment or we might say the tax is 6.1 cents on each dollar of assessment. But the expression *mills per dollar* is normally used. A mill is $\frac{1}{1000}$ of a dollar or $\frac{1}{10}$ of a cent.

$$\$1 = 1000 \text{ mills}$$
$$\$0.061 = 0.061 \times 1000 \text{ mills}$$
$$\$0.061 = 61 \text{ mills}$$

The tax rate is 61 mills per dollar of assessed value.

Some Provincial Responsibilities

Reproduced by permission from "How Canadians Govern Themselves", by Joseph Schull, Information Canada, Ottawa, 1971.

Obtain a provincial budget and make a circle graph showing the proportion of the budget spent on education, public works and other major expenditures.

Preparing Assessment Roll

The municipal assessor assesses not only residential property but all property in the municipality. Assessment notices are sent to all owners of property or land. If the owner feels the assessment is not fair he may appeal to the Court of Revision for an adjustment. The court hears the case and either upholds or alters the assessment.

After the assessor presents the statement of assessed values to council, the council debates, then decides on the tax rate. Study the simplified statement of assessed values of land and improvements.

Statement of Assessment
Whiterock, Manitoba

Type of Realty	*Land*	*Improvements*
1. Residential	$13,345,510	$38,786,070
2. Small Holdings	3,690,175	4,901,190
3. Agricultural	3,141,165	2,631,240
4. Commercial	2,479,690	5,700,881
5. Manufacturing	1,141,000	9,150,500
6. Utilities	256,050	3,550,760
7. Hydro Power		
Buildings		45,700
Land	150,000	
Poles		1,755,060
Machinery and Equipment		1,608,500
Total	$24,203,590	$68,129,901

Whiterock Council may decide to follow the policy of some provinces and not tax agricultural improvements.

EXERCISE 10·1

1. Name some reasons property value may change.

2. What do you think is the derivation of the term *mill*? Refer to the library, dictionary or encyclopedia.

3. What is the dictionary meaning of *assess*?

4. Obtain from your municipality an annual report showing the expenditures and revenues budget for the current year. Determine from the report the real property assessment. Can you determine the estimated revenue from real property tax? Can you determine how the tax rate was calculated? (Not all annual reports show a breakdown which enables a reader to calculate the tax rate.) Make a report on as many items as you can from the annual report. Keep the report for reference later.

5. From the annual report of your municipality, determine what services are provided. Show as a bar graph the amounts spent on each of the services.

6. Calculate the tax rate in Elmsville if the real property assessment is $45,300,000 and the budgeted tax expected from this source is $2,627,400.

7. What property assessment is necessary to raise $525,730 in taxes if the tax rate is $62\frac{1}{2}$ mills?

8. Millervale wishes to hold its tax rate at 59 mills. What assessment will it need in order to have a tax revenue of $773,450?

9. Queenstown has a total assessment of $1,432,500. Its mill rate is struck at $63\frac{3}{4}$ mills. What tax revenue will it have if only 94% of all taxes are collected?

Computer used in municipal finance department.

10. Following is a summary of a budget drafted by a town council.

Byetown Budget

Revenues		
Taxation		
real property	$5,122,000	
commercial and business	1,272,000	$6,394,000
Grants, Subsidies, etc.	750,400	
Per Capita Grants	152,125	902,525
Licences, Permits		141,300
Fines, Fees		97,500
Debenture Charges recovered		83,000
Others		51,200
		$7,669,525
Expenditures		
General municipal		356,500
Education		3,840,500
Police, Fire		720,810
Public Works, Sanitation, Garbage Removal, etc.		1,060,815
Hospitals, Nursing Homes, Social Welfare		552,700
Parks and Recreation		198,700
Library		150,000
Debt Service Cost and Reserves		641,700
Others		147,800
		$7,669,525

(a) Calculate the percent (use one decimal place) that each figure in the breakdown is of the revenue. Draw a circle graph which you could use to present the facts to the public.
(b) Repeat (a) for the items in the expenditures.
(c) Can you represent the parts on each circle graph as the *parts of each dollar* in revenue or expenditure. Explain.

11. Refer to the statement of assessment values for Whiterock. If the tax rate in Whiterock is $60\frac{1}{2}$ mills find the tax revenue (Whiterock does not tax farm buildings).

10·2 Paying Taxes

After the municipal council has set the tax rate, clerks in the office prepare and mail tax notices to property owners. Often bylaws make provisions for *discounts* if payments are made before a certain date or for *penalties* if payment of taxes is not made by a certain date. Some municipalities make provisions for payment in *instalments with discounts*. The example and exercises will illustrate various schemes for tax payment. Notice that assessment does not necessarily approximate sales value.

To calculate the tax which a property owner is to pay we use formula (1).

$$\text{tax} = \text{assessment} \times \text{rate}$$

Example 1

Mr. Smiley receives a tax notice for his house and property. The assessment is $16,200 and the tax rate is $55\frac{1}{2}$ mills. The tax notice indicates his taxes are $899.10. Provisions are made for a 1% discount on taxes paid before May 15 and $\frac{1}{2}$ of 1% discount on taxes paid between May 15 and July 15. If Mr. Smiley paid $500 on May 1 and the balance on June 1, what amount did he pay in taxes? How much did he save by paying as he did in instalments?

Solution

To check tax levy:

$$55\tfrac{1}{2} \text{ mills} = .0555 \text{ dollars}$$

$$\text{tax} = 16{,}200 \times .0555$$

$$= \$899.10$$

Therefore, the calculation of the tax is correct. Since he paid $500 this is 99% of the amount that is credited him (1% discount). Therefore,

$$\frac{500}{x} = \frac{99}{100}$$

$$x = \$505.05$$

On June 1 there is $899.10 – $505.05 = $394.05 still owing. Since he paid the balance before July 15 he need only pay 99.5% of the amount (0.5% discount).

$$99.5\% \text{ of } \$394.05 = \$392.08$$

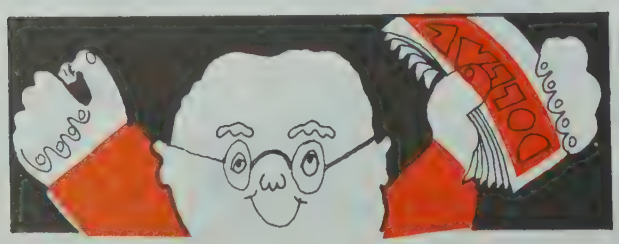
Therefore, Mr. Smiley pays

May 1	$500.00
June 1	392.08
Total	$892.08

Mr. Smiley pays $892.08 in taxes.

Tax	$899.10
Paid	892.08
Savings	$ 7.02

Mr. Smiley saved $7.02.



EXERCISE 10·2

1. Ian Scott built a new house. The total assessment on the property is $12,450. The tax rate is $48\frac{3}{4}$ mills. Find the tax on Mr. Scott's property.
2. The year after Mr. Scott built his house he added a greenhouse and garage to the property. The increase in assessment was $3,654. How much did his taxes go up due to the improvements? (Mill rate remained constant.)
3. The third year Mr. Scott was in his house the city council raised the tax rate to 51 mills. What was the tax bill Mr. Scott received that year?
4. City council also made provisions for discounts and penalties the third year Mr. Scott was in his house. Taxes paid before May 1 received a 5% discount and taxes paid after August 1 were assessed a 1% penalty. Mr. Scott paid $300 on April 15 and because of a layoff at work he did not pay the balance until October 1. Calculate the amount Mr. Scott paid in taxes
5. The City of Humbarg declares by bylaw that taxes are due 30 days after notice. After the 30 days a 3% penalty on the unpaid amount is assessed. Calculate the amount of taxes Mrs. Hollis would pay if she failed to meet the 30 day provision. Mill rate is 52. Land assessment is $2,000 and improvement assessment is $8,900.
6. Refer to the annual report produced by your municipality (see question 4, Exercise 10·1). What is the tax rate for your municipality?
7. If you live in a house owned by your family, ask your father or mother for the assessed value of the property. Then using the mill rate from question 6 calculate the tax on the property.
8. Albert Gauthier buys a home for $31,450 in Clarkesville, where homes are assessed at 35% of their market value. Find the tax bill when the tax rate is 71 mills. In Clarkesville, 62% of the taxes go to schools and education.
   (a) What did Mr. Gauthier pay for school taxes?
   (b) Mr. Gauthier has three children in public schools. What is the price per child that Mr. Gauthier pays for their education?

10·3 Local Improvement Taxes

In some developments of a city or town, charges for sidewalks, paved streets, curbs, watermains, sewers, and similar improvements are made against each parcel or lot of land fronting on the improved area. These improvements are called *local improvements.* The charges are called *local improvement taxes.* The taxes collected are used to pay for these improvements over a period of time. Therefore, the local improvement taxes for a specific improvement are spread over a period of years. This tax is usually charged on a per foot frontage – *frontage* is the length of the lot facing on the area of improvement.

Example 1

Fifth Street, between 40th and 48th Avenue, has curbs and pavement laid. The frontage charge is $1.35 per foot. Mr. Neil has a 55 foot lot facing on Fifth Street. He has the option of paying the full amount in the first year or of having $17.35 added to his tax bill each year for five years.

(a) What was his cost for the local improvement?
(b) If he chooses the deferred payment method how much interest will he pay?

Solution

(a) The cost of the local improvements is $74.25

$$\text{cost} = \text{frontage} \times \text{rate}$$

$$= 55 \times 1.35$$
$$= \$74.25$$

(b) The amount he pays under the deferred payment plan is:

$$\$17.35 \times 5 = \$86.75$$

The interest is the difference between the deferred payment and the cost:

$$\$86.75 - \$74.25 = \$12.50$$

EXERCISE 10·3

1. Make a survey of 5 houses in the neighbourhood of your school or in some portion of your community to find the assessment value of the 5 houses. Calculate the tax on each house.

Before you call on the houses, prepare what you are going to say to the homeowners. You will want to answer questions that the homeowners will ask such as:

(a) purpose of survey?
(b) what use will be made of the data collected?
(c) will names of the owners or location of the property be revealed or used?
(d) does the homeowner have to give you the information?

Remember to be polite and well-mannered. If a homeowner does not want to give the information to you or if he does not have the information available, do not press. Thank him politely and go on your way.

Hint! A homeowner may not know the assessment of his property but he may know his tax bill. Will this be sufficient? After the survey can you answer this question: Is assessment value close to the actual value in this neighbourhood?

In making the survey of 5 houses obtain information about local improvement taxes. What questions will you ask?

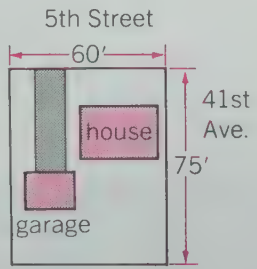
Determine:

(a) whether or not local improvement taxes are common,
(b) the average cost of the local improvement tax to each home owner,
(c) if the property owner was aware when he bought the property that there would be a local improvement tax.

2. If you live in a privately owned home, endeavour to find out whether or not there is a local improvement tax charged to the lot your house is on. Determine the local improvement tax and why the tax was imposed.

3. Mr. Charles owns a 65-foot lot on a street on which there is a frontage charge of $1.10 for paving and curbs, and a charge of $0.35 a foot for underground wiring. Mr. Charles elects to pay on the deferred plan of $0.31 a foot for 5 years.
   (a) What was his share of the cost of the improvements?
   (b) What interest did he pay on the deferred plan?
   (c) What was the apparent interest (simple) rate?
   (d) What was the real interest rate? (Refer to chapter on instalment buying.)

4. City council levies a frontage tax of $1.60 per foot on all property in a certain subdivision for new and improved facilities. Ray Jones owns the corner lot shown at the right.
   (a) If there is no reduction in rate for corner lots, how much improvement tax will he pay for these improvements?
   (b) Do you think this is a fair way to tax corner lots for local improvement? Explain.



   (c) Find out how your city charges for local improvements on corner lots.

10·4 Business Tax

Business Tax Gets Approval

The finance committee Tuesday approved the passing of a business tax in lieu of a machinery tax in Calgary.

The business tax will produce $628,091 in revenue whereas a tax on all industrial and commercial machinery would have produced $1,021,320, or $393,229 more.

The business tax is based on a percentage of the rental price for the premises occupied by a business.

To understand the newspaper clipping we need to be aware of *Business Tax*.

All property is assessed. Some properties are exempt from taxation—schools and churches are two examples—while others may have a different rate from residential property. Business taxes are usually assessed in addition to the residential and commercial assessment. Basically there are three methods of applying a business tax:

1. a percentage of the commercial property assessment,
2. rate per square foot of the business premises,
3. percentage of the current annual rental value.

The percentage or rate will vary with the type of business. If the premises are used for both residential and business purposes, then the total assessment is divided between the two on the basis of the proportion of the premises used for each. The examples and exercises will illustrate the various points discussed in this section.

Example 1

A certain building assessed at $31,000 is on a lot assessed at $8,500. The building houses a business complex on the first floor and residential apartments on the second and third floors. The apportionment of the assessments is as follows:

	Land	Building	Tax rate
Residential	40%	50%	58.6 mills
Commercial	60%	50%	61.4 mills

The business tax is 33% of the commercial tax. Find the residential, commercial, and business taxes.

Solution

(a) Residential tax assessment is:

$$\begin{aligned} 40\% \text{ of } \$8{,}500 &= \$\ 3{,}400 \\ 50\% \text{ of } \$31{,}000 &= \$15{,}500 \\ \text{Total assessment} &= \$18{,}900 \end{aligned}$$

Residential tax = $\$18{,}900 \times 0.0586 = \$1{,}107.54$.

(b) Commercial tax assessment is:

$$\begin{aligned} 60\% \text{ of } \$8{,}500 &= \$\ 5{,}100 \\ 50\% \text{ of } \$31{,}000 &= \$15{,}500 \\ \text{Total assessment} &= \$20{,}600 \end{aligned}$$

Commercial tax = $\$20{,}600 \times .0614 = \$1{,}264.84$

(c) Business tax is 33% of the commercial tax:

$\$1{,}264.84 \times .33 = \$417.39$.

Business tax is $417.39.

EXERCISE 10·4

1. Investigate and report to the class which lands or properties in your municipality
   (a) are exempt from taxation;
   (b) have rates different from residential—and what the different rates are.

   Find what scheme for business and commercial taxes is used.

2. Choose teams to debate the issues below. Research and prepare for the debate.
   (a) The sales tax should be increased to cover the increased cost of education rather than an increase in residential property tax.
   (b) Rather than an increase in general property tax, business and commercial tax should be increased to cover the increased cost of _____. (Insert in the blank the name of any project currently being planned or built in your community—e.g. downtown mall, expressway, etc.)

3. A business firm rents certain premises for $1,150 a month. The business tax charged by the city is 28% of the rental price. What is the amount of the annual tax?

4. A building has a business area on the first floor and residential area on the second floor. The assessed value of the building is $6,500. The land which includes a parking lot is assessed at $2,300. The business tax is 33% of the commercial tax. The division of the assessment is as follows:

	Land	Building	Tax rate
Commercial	80%	50%	70 mills
Residential	20%	50%	51.4 mills

   (a) Find the total tax bill on the real property.
   (b) What percent of the total tax bill is the residential tax?

5. The ABC Toy Store occupies 7200 square feet of space. The annual business tax rate is 19 cents per square foot. What is the annual tax on the business?

6. The M.N. Company built a new commercial building on a large city lot. The land is assessed at $8,500 while the building is assessed at $36,350. The commercial tax rate is 57.8 mills while the business tax is 34% of the commercial tax.
   (a) Construct a flow chart to find the total tax M.N. Company pays annually.
   (b) Find the tax M.N. Company pays annually.
   (c) There is a 5% discount on taxes paid before March 15. The company pays $2000 before the discount deadline and the balance after the deadline. Use a flow chart continuing from the one in part (a) to show the amount in taxes the company pays.
   (d) Find the amount of taxes M.N. Company pays.

7. Discuss in class the meaning of the news item introducing this section.

10·5 Theory in Practice

Figures 10-1 and 10-2 show the front view of the Notice of Assessment on a home in the city of Newtonville for 1973. Figure 10-3 shows the reverse portion of Figure 10-1. Obtain similar copies for homes in your city.

The City of Newtonville

NOTICE OF ASSESSMENT
FOR YEAR 1973

CODING
121

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

ROLL NUMBER		
38	1110	2

VALUE OF LOT OR PARCEL WITHOUT IMPROVEMENTS	VALUE OF BUILDING AND OTHER IMPROVEMENTS	VALUE OF MACHINERY AND OTHER EQUIPMENT
1930	6960	

TOTAL TAXABLE VALUE	FINAL DATE OF APPEAL
8890	

VALUE OF PROPERTY	
PUBLIC SCHOOL SUPPORTER	SEPARATE SCHOOL SUPPORTER
8890	

Figure 10-1

The City of Newtonville

PAYMENT ON ACCOUNT OF 1973 PROPERTY TAX

ROLL NUMBER		
38	1110	2

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

PRINT CLEARLY

AMOUNT
$

MAKE CHEQUES PAYABLE TO THE CITY OF NEWTONVILLE

Figure 10-2

TO:
THE ASSESSOR OF THE CITY OF NEWTONVILLE.
SIR: I HEREBY APPEAL TO THE COURT OF REVISION AGAINST THE ASSESSMENT OF

_____

_____

(HERE STATE PROPERTY AFFECTED. ROLL NUMBER)
ADDRESS OR LEGAL DESCRIPTION
ON THE FOLLOWING GROUNDS

(HERE STATE GROUNDS OF APPEAL)

DATED THIS_____ DAY OF JANUARY, 1973

SIGNATURE
OF APPELLANT _____

Figure 10-3

Figures 10-4 and 10-5 show the two sides of the insert enclosed with the Notice of Assessment.

ASSESSMENT CARD CODING

(1) GENERAL ROLL
1. AGRICULTURAL
2. RESIDENTIAL
3. COMMERCIAL
4. SPECIAL PURPOSES BUILDINGS
5. INDUSTRIAL
6. APARTMENTS
7. RAILWAYS
8. PARKING LOTS
9. SEC 459 DEV.

0. Vacant
1. Single Dwelling
2. Duplex
3. Triplex
4. Rooming or Boarding Houses
9. Miscellaneous

EXAMPLE:-
Assessment Card Code Entry

1	2	2

General Roll ↓ Residential — Duplex

Figure 10-4

The City of Newtonville

Payment on Account of 1973 Property Tax

Payment on account of property taxes for the year 1973 will be accepted from January 2nd, 1973 to February 28th, 1973 inclusive, and interest at the rate of five per cent (5%) per annum from date of payment to June 30th, 1973, will be allowed on all such payments.

All 1973 taxes paid before January 2nd, 1973, shall bear interest at the rate of five per cent (5%) per annum from January 2nd, 1973 to and including June 30th, 1973.

Payments, including interest, will be credited directly to the account and will appear as a credit on the tax statement, when issued.

Payments may be made in any amount but should not exceed the amount of the 1972 taxes.

R. D. HAMILTON
Tax Collector

Figure 10-5

Figures 10-6, 10-7 and 10-8 show two portions of the Tax Statement and its reverse side. Obtain similar copies for homes in your city.

TERMS OF PAYMENT, etc.

Payments on Account Accepted At Any Time
(Subject to Penalties and Interest Charges)

No penalty shall be added to any amount of current taxes paid on or before June 30th, 1973.

To all current taxes remaining unpaid after June 30th, 1973, penalties will be added in the following manner:

On July 1st, 1973, there will be added a penalty of 2% of the unpaid 1973 taxes.

On September 1st, 1973, the penalty will be increased to 3% of the unpaid 1973 taxes.

On November 1st, 1973, the penalty will be further increased to 4% of the unpaid 1973 taxes.

ARREARS:-

Interest at ⅔ of 1% of the total tax arrears and accrued penalties will be added on the 1st day of each and every month of 1974 that the arrears remain unpaid.

All lands in arrears for the previous year are liable to proceedings under the Tax Recovery Act of 1938 (and amendments).

Receipts issued in acknowledgement of a cheque or other negotiable instrument shall be valid only when the amount of such cheque or instrument has been collected by The City of Newtonville.

All cheques should be made payable to The City of Newtonville. Notify Tax Department of any error in name, address or description of property.

R. D. HAMILTON
Tax Collector

Figure 10-6

The City of Newtonville

038	01110	2	3836 VANCOUVER CR NW	121
ROLL NUMBER			ADDRESS	CODING

HOW YOUR 1973 TAXES ARE ALLOCATED BASED ON 65.60 MILLS

1973 TAX STATEMENT

NO PENALTY IF PAID ON OR BEFORE
30 JUNE 1973

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

MUNICIPAL TAXES	205.09
EDUCATION TAXES	378.09
	5.70
TOTAL TAX PAYABLE	588.88

IF ELIGIBLE FOR PROVINCIAL TAX DISCOUNT SUBMIT APPLICATION AND PAY NET TAX: 538.88

Figure 10-7

LOCAL IMPROVEMENT AND SPECIAL TAXES	
DESCRIPTION OF LEVY	AMOUNT
WATER FTGE. TAX IMP.	5.70
TOTAL	5.70

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

City of Newtonville

111541 8	
MORTGAGE CO. REFERENCE	
038 01110 2	
ROLL NUMBER	
TOTAL 1973 TAXES	588.88
IF ELIGIBLE FOR PROVINCIAL TAX DISCOUNT	538.88
PART PAYMENT	

Figure 10-8

EXERCISE 10·5

1. Shown here are the 1972 Notice of Assessment and Tax Statement.

   (a) What was the mill rate change from 1972 to 1973? What percent increase is shown?

   (b) Notice the change in assessment. The change was due to the rumpus room and auxiliary rooms built in the lower level of the house. What assessment increase resulted?

   (c) The taxes were paid on February 15, 1972. What amount was necessary to pay the tax in full. (The home owners were eligible for the Provincial Tax discount.)

The City of Newtonville

121

NOTICE OF ASSESSMENT
FOR YEAR 1972

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

ROLL NUMBER		
38	1110	2

VALUE OF LOT OR PARCEL WITHOUT IMPROVEMENTS	VALUE OF BUILDING AND OTHER IMPROVEMENTS	VALUE OF MACHINERY AND OTHER EQUIPMENT
1930	6530	

TOTAL TAXABLE VALUE	FINAL DATE OF APPEAL
8460	29 JAN 72

VALUE OF PROPERTY	
PUBLIC SCHOOL SUPPORTER	SEPARATE SCHOOL SUPPORTER
8460	

The City of Newtonville

038	01110	2	3836 VANCOUVER CR N W	
ROLL NUMBER			ADDRESS	

HOW YOUR 1972 TAXES ARE ALLOCATED BASED ON 60.60 MILLS

1972 TAX STATEMENT

NO PENALTY IF PAID ON OR BEFORE

28 JUN 72

Robert Andrew Jones
Joanne Valerie Jones
3836 Vancouver Cres N W

MUNICIPAL TAXES	191.33
EDUCATION TAXES	321.35
	5.70
TOTAL TAX PAYABLE	518.38

IF ELIGIBLE FOR PROVINCIAL TAX DISCOUNT SUBMIT APPLICATION AND PAY NET TAX 468.38

2. Discuss the various parts of the Notice of Assessment and the insert enclosed with it. In many places a property owner declares himself a Public School Supporter (usually non-Roman Catholic School Supporter) or Separate School Supporter (usually Roman Catholic School Supporter).

3. Discuss the various parts on the Tax Statement.

4. Homes are often purchased jointly in the names of the husband and wife. Robert Andrew Jones, Joanne Valerie Jones own the property shown. Their home is mortgaged. Mr. and Mrs. Jones paid $100 on the 1973 taxes on December 28, 1972. On February 15, 1973 they paid another $100. They paid the balance as shown:

   July 10, $100, October 1, $100; March 10, 1974 the balance.

   Calculate the total amount they paid. (Mr. and Mrs. Jones were eligible for the home owner discount.)

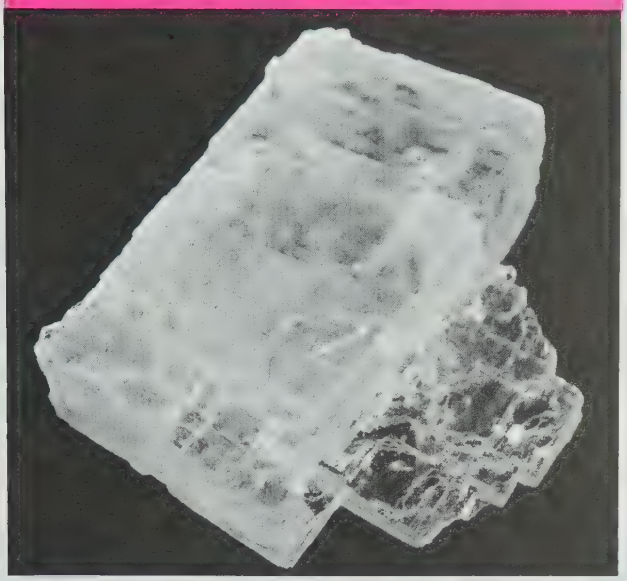
5. The provincial government gives the home owner discount (Provincial Tax Discount) to the home owner who lives in his own house, to help relieve his tax load. The provincial government then reimburses the municipal government this amount. Debate this practice as an effort to help the home owner in a time of high taxes.

6. For what reasons might a home owner appeal the assessment?

Using the Library

1. *The Giant Golden Book of Mathematics.* Read pages 30-32. What is the Fibonacci ratio? See if you can find examples of the Fibonacci ratio in nature. How is the Fibonacci ratio related to the Golden Section?

2. *Mathematics Enrichment.* Read pages 120-131. Prepare as many illustrations as you can of mathematics in nature. Research the *Curve of Pursuit*.



How is this an example of math in nature?

REVIEW EXERCISE 10

1. In Parksville the total assessment is $5,600,000. The council plans to obtain 80% of its total budget of $444,500 from real property taxes. What mill rate (tax rate) must the council strike (set)?

2. Hollowridge Town Council makes provision for taxes to be paid in three equal instalments on March 1, May 1 and July 1. A discount of 5% is allowed on the March 1 payment, 3% on the May 1 payment and 1% on the July 1 payment. A taxpayer with a tax bill of $654 wishes to pay in three equal instalments. How much does he save in discounts?

3. Mr. Allan owns a home with combined land and house value of $35,000 on a 75-foot lot in a city which assesses property at 30% of its market value. The tax rate is $51\frac{1}{2}$ mills. There is a frontage tax of 15 cents per foot per year for 10 years. Mr. Allan may pay off the frontage tax in a lump payment of $1.20 per foot.

   (a) What is Mr. Allan's tax bill excluding frontage tax? including frontage tax?

   (b) What is the interest charge on the deferred payment basis?

   (c) What is the apparent interest rate?

   (d) What is the real interest rate?

4. A city council approves a bylaw charging a business tax of 22.5 cents per square foot of business space occupied, in lieu of a machinery tax of 12 cents per dollar of assessed value of the machinery. The Trailer Hitch Manufacturing and Sales Company occupies 5600 square feet of space and has installed machinery assessed at $11,500. Calculate the tax the Trailer Hitch Manufacturing and Sales Company would pay under the

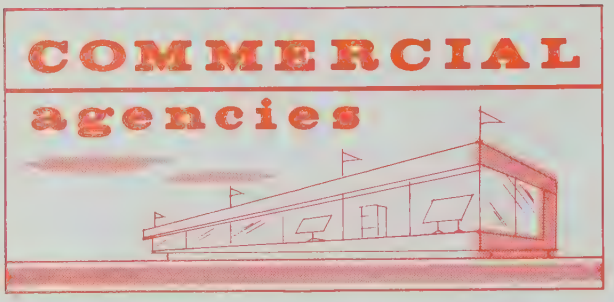
   (a) original tax scheme

   (b) new tax scheme.

   Which of the two schemes do you think the company would like better? Why?

   Hint! Think of an expanding business.

5. A city has a total property assessment of $450,000,000. It decides to raise $750,000 from property tax to improve medical facilities at a hospital. What is the mill rate necessary to raise this amount?



Commercial Agencies operates two lines of business. We have a retail store selling engineering and drafting equipment, of which a good portion is imported from such countries as U.S.A., England, Denmark, Germany and Japan. As we do our own customs clearing, we have supplied you with a sample that you could use in regard to working out cost of merchandise bought. We might mention that we would consider this a very simple sample, and if you would care to receive a more difficult one with quite a number of items listed on one invoice plus freight added as an expense, where the cost of each item has to be worked out on an expense, percentage basis, we shall be glad to supply you.

Commercial Agencies also operates an Ozalid or Dryprinting Department, also known as Blueline, and we have supplied you with two transactions as they occur daily in our business.

Dryprinting Department

Commercial Agencies operates an Ozalid or Dryprinting Department, where any drawings can be reproduced in any number required. Customers are charged, let us say, .07¢ per square foot run through the machine. It is also Government regulations to charge 12% sales tax on the completed job. Calculate these charges:

Problem No. 1:

A customer brings in 4 original drawings and wants 5 copies made of each drawing. Each drawing is 6 sq. ft.

Problem No. 2:

A customer brings in 3 original drawings, one is 6 sq. ft, one is 5 sq. ft, one is 10 sq. ft. He wants 3 copies made of each.

Store

Commercial Agencies operates a retail store selling engineering and drafting equipment of which a good portion is imported from various countries. Most invoices from suppliers outside of Canada are charged in US dollars. The shipments, if they are small, arrive at the Customs Office, Parcel Post Branch, and we have to clear them through Customs before we can receive them. We do our own clearing, and here is what you have to work out:

A. Convert the US currency into Canadian currency.
B. Establish the rate and amount of duty to be paid.
C. Work out the Sales Tax to be paid.

A. Find out from your newspaper or from a local bank the rate of exchange for US currency.
B. The rate of duty for engineering and drafting equipment is 9%.
C. The Federal Sales Tax is 12%.

Problem A

We received 5 Ruling Pens. The invoice reads:

5 Ruling Pens	Ea. US 5.00	$US 25.00

Problem B

Establish the cost of one Ruling Pen.

To establish a selling price it is usual to work with a markup of 40%. This is 40% of the selling price, not of the cost.

Problem C

Work out a selling price for each Ruling Pen.

Chapter Eleven
RADICALS AND EXPONENTS

11·1 Renewing Acquaintances

The irrational numbers we must deal with most frequently appear as radicals. Examples of radicals are $\sqrt{2}$, $\sqrt{7}$, $\sqrt[3]{3}$, $\sqrt[5]{5}$ and $\sqrt[5]{4}$. The square roots $\sqrt{2}$, $\sqrt{7}$, are called *second order radicals*. The general form of the second order radical is

$$\sqrt{n},\ n \in R,\ n \geq 0.$$

The radical $\sqrt{3a}$ is composed of radical sign, $\sqrt{\ }$, and the radicand, $3a$. Not all square roots are irrational. For example, $\sqrt{4} = 2$, therefore, $\sqrt{4}$ is rational. $\sqrt{6}$ is irrational since there exists no rational x such that $x^2 = 6$.

Example 1

Express $\sqrt{3^2}$ as a rational number without the radical sign.

Solution

$\sqrt{3^2} = 3$ since $3 > 0$

Note that $\sqrt{a}, a > 0$, is always positive. We call this positive quantity the principal square root of a.

The cube roots such as $\sqrt[3]{25}$ and $\sqrt[3]{15}$ are called *third order radicals*. The general form of the cube root is

$$\sqrt[3]{n},\ n \in R$$

3 is the index	$\sqrt{\ }$ is the radical sign
n is the radicand	$\sqrt[3]{n}$ is the radical

Notice in third order radicals the radicand may be negative, zero, or positive. This is in contrast to the second order radicals. We note that

$\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$, $\sqrt[3]{8} = 2$ and $\sqrt[3]{0} = 0$.

Example 2

Write the standard numeral for each.

(a) $\sqrt{64}$ (b) $\sqrt[3]{-27}$ (c) $\sqrt[5]{32}$ (d) $\sqrt[3]{-\frac{64}{125}}$

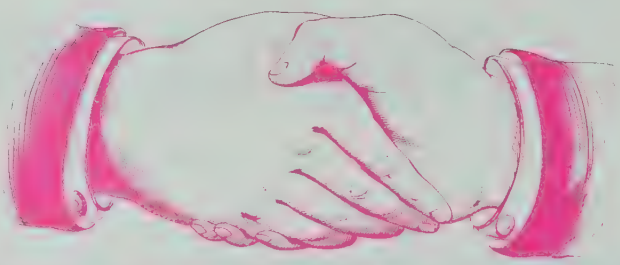
Solution

(a) $\sqrt{64} = 8$ (because $8^2 = 64$)

(b) $\sqrt[3]{-27} = -3$ (because $(-3)^3 = -27$)

(c) $\sqrt[5]{32} = 2$ (because $2^5 = 32$)

(d) $\sqrt[3]{-\frac{64}{125}} = -\frac{4}{5}$ (because $(-\frac{4}{5})^3 = -\frac{64}{125}$)



EXERCISE 11·1

1. Why does the definition of $\sqrt{b}$ include the condition that $b \geq 0$ when we are talking about the set of real numbers?

2. Identify the numbers which are *not* members of the set of real numbers. Tell why each you identify is *not* a member of the set R.

   (a) $\sqrt{3}$ (b) $\sqrt{9}$
   (c) $\sqrt{x}$ (d) $\sqrt{-4}$
   (e) $\sqrt{b}, b > 0$ (f) $\sqrt{6-3}$
   (g) $\sqrt{-2-(-4)}$ (h) $\sqrt{-9}$
   (i) $\sqrt{a-b},\ a > b$ (j) $\sqrt{m-n}, m > n$

3. Write the standard numeral for each.

   (a) $\sqrt[3]{125}$ (b) $\sqrt{81}$
   (c) $\sqrt[4]{256}$ (d) $\sqrt[6]{64}$
   (e) $\sqrt{\frac{1}{9}}$ (f) $\sqrt[3]{-\frac{8}{27}}$
   (g) $\sqrt[3]{\frac{729}{1000}}$ (h) $\sqrt[3]{-\frac{1}{216}}$
   (i) $\sqrt[4]{\frac{10000}{81}}$ (j) $\sqrt{1.44}$
   (k) $\sqrt[3]{1.728}$ (l) $\sqrt{2.25}$
   (m) $\sqrt{(5x)^2}, x \geq 0$ (n) $\sqrt{a^2b^4}, a, b \geq 0$
   (o) $\sqrt{25c^2}, c \geq 0$ (p) $\sqrt[3]{343d^3}$

11·2 The Product and Quotient Properties of Square Roots

Most of the work with radicals in this book will be with square roots although cube roots and roots of higher orders will be used at times.

One method of finding the square root of a large number is to determine the prime factors of the number then express the number as the product of powers and finally take the square roots of the powers.

Example 1

Evaluate $\sqrt{1764}$.

Solution

Find the prime factors of 1764.

$$\begin{array}{r|l} 2 & 1764 \\ 2 & 882 \\ 3 & 441 \\ 3 & 147 \\ 7 & 49 \\ & 7 \end{array}$$

Divide the number and each successive quotient by a prime number until the quotient is prime.

Now express the number as the product of powers (group the factors):

$$1764 = 2^2 \times 3^2 \times 7^2$$

$$\therefore \quad \sqrt{1764} = \sqrt{2^2 \times 3^2 \times 7^2}$$
$$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{7^2}$$
$$= 2 \times 3 \times 7$$
$$= 42.$$

The solution in Example 1 uses the *product property of square roots.*

The Product Property of Square Roots:
$\sqrt{nm} = \sqrt{n} \times \sqrt{m}$, $n, m \in R, n \geq 0, m \geq 0$

This property allows us to multiply radicals. You learned earlier that the set of real numbers has the properties of the set of rational numbers. Can you name the properties of the rational numbers?

What does the introduction of the irrationals allow you to do that you would not do in the set of rationals?

The set of real numbers is closed for multiplication. We will want to know how to multiply two irrational numbers. The product property tells us:

$$\sqrt{n} \times \sqrt{m} = \sqrt{n \times m}$$

This property is illustrated in the next example.

Example 2

Simplify:

(a) $2\sqrt{3} \times \sqrt{6}$ (b) $m\sqrt{a} \times n\sqrt{b}$.

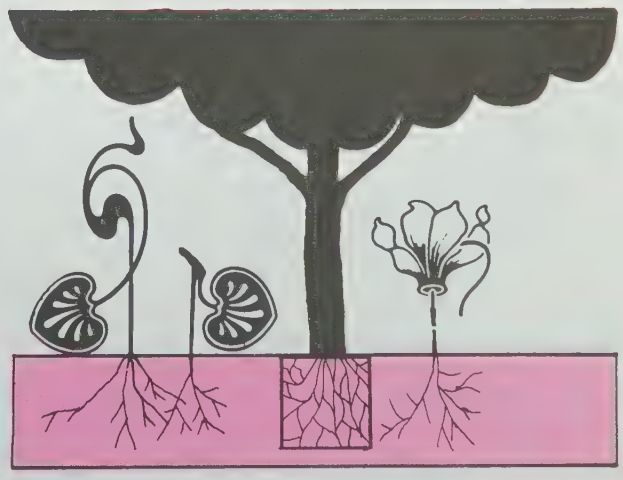
Solution

(a)
$$\begin{aligned} 2\sqrt{3} \times \sqrt{6} &= 2(\sqrt{3} \times \sqrt{6}) \\ &= 2(\sqrt{3 \times 6}) \\ &= 2(\sqrt{18}) \\ &= 2(\sqrt{3 \times 3 \times 2}) \\ &= 2 \times 3\sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

(b)
$$\begin{aligned} m\sqrt{a} \times n\sqrt{b} &= m \times n \times \sqrt{a} \times \sqrt{b} \quad \text{(Why?)} \\ &= mn\sqrt{ab} \end{aligned}$$

Similarly, we can assume the set of real numbers (rationals and irrationals) is closed for division. Can you guess the *Quotient Property of Square Roots*?

The Quotient Property of Square Roots:
$\sqrt{\frac{n}{m}} = \frac{\sqrt{n}}{\sqrt{m}}$, $n, m \in R, n \geq 0, m > 0$



Example 3

Evaluate $\sqrt{28} \div \sqrt{63}$.

Solution

$$\sqrt{28} \div \sqrt{63} = \frac{\sqrt{28}}{\sqrt{63}} = \sqrt{\frac{4 \times 7}{9 \times 7}}$$
$$= \sqrt{\tfrac{4}{9}}$$
$$= \tfrac{2}{3}$$

Example 4

Evaluate $\sqrt{\frac{1089}{1225}}$.

Solution

$$\sqrt{\frac{1089}{1225}} = \frac{\sqrt{1089}}{\sqrt{1225}} = \frac{\sqrt{3^2 \times 11^2}}{\sqrt{5^2 \times 7^2}} = \frac{3 \times 11}{5 \times 7} = \tfrac{33}{35}$$

3)1089	5)1225
3)363	5)245
11)121	7)49
11	7

The two properties we have used in this section give us the power to express real numbers in different forms. We can write $\sqrt{32}$ in the equivalent form $\sqrt{16 \times 2}$ which in turn is equivalent to $4\sqrt{2}$. The numeral $\sqrt{32}$ is called a *pure* or *entire radical*. The numeral $4\sqrt{2}$ is called a *mixed radical*.

Example 5

Write each as a mixed radical.

(a) $\sqrt{252}$ (b) $\sqrt{45a^2b^4}$

Solution

(a) $\sqrt{252} = \sqrt{2^2 \times 3^2 \times 7}$
$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{7}$
$= 6\sqrt{7}$

(b) $\sqrt{45a^2b^4} = \sqrt{5 \times 3^2 \times a^2 \times b^4}$
$= \sqrt{5} \times \sqrt{3^2} \times \sqrt{a^2} \times \sqrt{b^4}$
$= \sqrt{5} \times 3 \times a \times b^2$
$= 3ab^2\sqrt{5}$

Example 6

Write each as a pure radical.

(a) $5\sqrt{11}$ (b) $16x\sqrt{7y}$

Solution

(a) $5\sqrt{11} = \sqrt{5^2} \times \sqrt{11}$
$= \sqrt{25 \times 11}$
$= \sqrt{275}$

(b) $16x\sqrt{7y} = \sqrt{16^2} \times \sqrt{x^2} \times \sqrt{7y}$
$= \sqrt{256 \times x^2 \times 7y}$
$= \sqrt{1792x^2y}$

You see? You forgot to take the square root of the pressure.

EXERCISE 11·2

Evaluate each expression using the method illustrated in this section.

1. $\sqrt{900}$
2. $\sqrt{1296}$
3. $\sqrt{7056}$
4. $\sqrt{\frac{484}{1089}}$
5. $\sqrt{\frac{1521}{64}}$
6. $\sqrt{\frac{676}{196}}$
7. $\sqrt{\frac{169}{576}}$
8. $\sqrt{\frac{6889}{1296}}$
9. $\sqrt{\frac{x^5y^7}{x^3y^2}}$

Write each as a mixed radical.

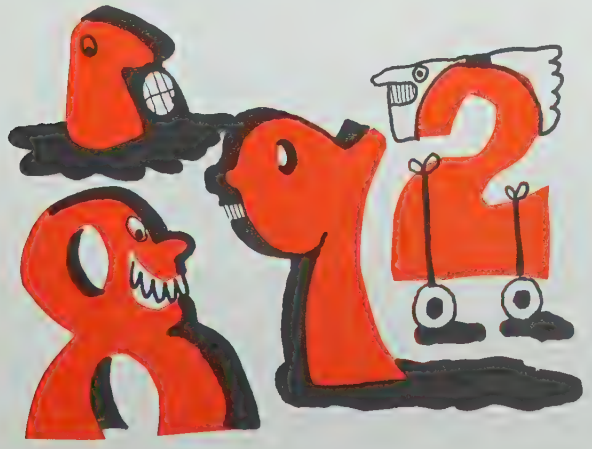
10. $\sqrt{32x^2}$
11. $\sqrt{84a^2b^2}$
12. $\sqrt{11m^2n^4}$
13. $\sqrt{58m^2n^5}$
14. $\sqrt{13013}$
15. $\sqrt{5^3m^3y^3}$
16. $\sqrt{(a-b)^2b}$
17. $\sqrt{\frac{48}{49}}$
18. $\sqrt{(x^2+4x+4)y^3}$
19. $\sqrt{3(2a+b)^2}$
20. $\sqrt{b(x^2+14x+49)}$
21. $\sqrt{(a^2-2a+1)c}$

Solve each for the variable indicated. The set of real numbers is the domain of each variable. Explain each instance in which the solution set is the empty set.

22. $4x^2 = 9$
23. $9y^2 = 48$
24. $13y^2 = 169$
25. $16y^2 + 8 = 0$
26. $81 - 5y^2 = 0$
27. $169m^2 - 64 = 0$
28. $25n^2 + 4 = 0$
29. $32 + 18y^2 = 0$
30. $27m^2 - 125 = 0$
31. $4t^2 + 8 = 24$

Write each as a pure radical.

32. $3a\sqrt{bc}$
33. $14\sqrt{b}$
34. $\frac{1}{3}\sqrt{mn}$
35. $15x^2\sqrt{yz}$
36. $9xyz^2\sqrt{3xy}$
37. $11a^2b^3\sqrt{2ab}$
38. $\frac{2}{5}\sqrt{\frac{a}{5}}$
39. $\frac{3}{8}x^2\sqrt{\frac{3}{8}ax}$



11·3 Approximations of Irrational Numbers

To say that the measure of a segment is $\sqrt{3}$ units is quite impractical. How would you measure such a segment? You will recall from earlier work with irrationals that a characteristic of this set is that no member can be expressed as a repeating decimal: terminating or nonterminating. *Any attempt to express $\sqrt{3}$ as a rational number either as a decimal or a fraction will only result in an approximation.* For all practical purposes the approximations are accurate enough. There are certain methods that allow us to get the results more easily than by other methods.

Previously you found square roots by Newton's method or the slide rule. Tables are provided in most texts. Refer to the table of Powers, Roots and Reciprocals on page 263. Below is a portion of the table.

Table 1

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526

There are several things to notice about this table. First, the domain for n is $1 \leq n \leq 100$. Second, values for $\sqrt{n}$ are approximate to 3 decimal places. We say the value of $\sqrt{3}$ is *correct* to 3 decimal places. The examples that follow illustrate how to use this table.

Example 1

Use the table to evaluate each correct to 3 decimal places.

(a) 47^2 (b) 8.2^2 (c) $\sqrt{50}$ (d) $\sqrt{2304}$

Solution

(a) To evaluate 47^2 we find 47 in the column headed n. Then we go across that row to the column headed n^2. The value of 47^2 is 2209.

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204

(b) To evaluate 8.2^2 we find 82 in the column headed n. Then we go across to the column headed n^2. We find the number 6724.

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120

But we want 8.2^2 not 82^2. We know the answer is a little more than 64; therefore, the decimal must be placed correctly: 67.24. Hence, $8.2^2 = 67.24$.

(c) To find $\sqrt{50}$ we go down the n column to 50. Then we go across to the column marked $\sqrt{n}$. Here we find the number 7.071. $\sqrt{50} = 7.071$ (correct to 3 decimal places).

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200

(d) Since 2304 is greater than 100 we must use a method other than that in (c). Therefore, we locate 2304 or the number closest to it in the table in the column headed n^2. Then we go left in this row to the column headed n. We find the number 48. Therefore, $\sqrt{2304} = 48$.

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204

Rationalizing the Denominator

We can now use the table of squares and square roots to evaluate approximately some of the irrational expressions with which we have been working. For instance, if you were asked to find an approximate value of $\frac{\sqrt{39}}{\sqrt{2}}$ you might proceed in this way:

$\sqrt{39} = 6.2450$ (correct to 4 decimal places)

$\sqrt{2} = 1.4142.$

Therefore, $\frac{\sqrt{39}}{\sqrt{2}} = \frac{6.2450}{1.4142}$

$= 4.416$ (correct to 3 decimal places)

But this involved dividing $1.4142\overline{)6.2450}$ which generally most people can do but find tedious. A method that eliminates this long division in which the divisor is a four or five digit number is shown below.

Example 2

Evaluate $\frac{\sqrt{39}}{\sqrt{2}}$ correct to 3 decimal places.

Solution

$\frac{\sqrt{39}}{\sqrt{2}} = \frac{\sqrt{39}}{\sqrt{2}} \times \frac{1}{1}$ Multiplicative identity

$= \frac{\sqrt{39}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $\frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{1}$, replacement

$= \frac{\sqrt{39 \times 2}}{\sqrt{2 \times 2}}$ Product property of square roots

$= \frac{\sqrt{78}}{2}$ This is called the *rationalized form* of $\frac{\sqrt{39}}{\sqrt{2}}$

$= \frac{8.832}{2}$

$= 4.416$ Correct to 3 decimal places

The process of transforming a fraction with an irrational number in the denominator such as in $\frac{5}{\sqrt{a}}$ or $\sqrt{\frac{7}{a}}$, $a \neq 0$, into an equivalent fraction with a rational number in the denominator $\frac{5\sqrt{a}}{a}$ or $\frac{\sqrt{7a}}{a}$, is called *rationalizing the denominator.*

The word *rationalize* means *to make rational.* Hence, to *rationalize the denominator* means to make the denominator a rational number.

An expression containing a radical is said to be in simplest form when:

1. the denominator is a rational number
2. the radicand contains no factor for which the particular root can be taken
3. the radical is of lowest terms
4. all rational numbers are simplified.

Example 3

Rationalize and simplify:

(a) $\frac{3\sqrt{7}}{\sqrt{3}}$ (b) $\frac{\sqrt{45}}{\sqrt{12}}$ (c) $\frac{ab^2\sqrt{a}}{\sqrt{b}}$.

Solution

(a) $\frac{3\sqrt{7}}{\sqrt{3}} = \frac{3\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ Why?

$= \frac{3\sqrt{7\times 3}}{\sqrt{3\times 3}}$ Why?

$= \frac{3\sqrt{21}}{3}$

$= \sqrt{21}$

(b) $\frac{\sqrt{45}}{\sqrt{12}} = \frac{\sqrt{45}}{\sqrt{12}} \times \frac{\sqrt{12}}{\sqrt{12}}$ Why?

$= \frac{\sqrt{45\times 12}}{12}$

$= \frac{\sqrt{3\times 3\times 5\times 2\times 2\times 3}}{12}$

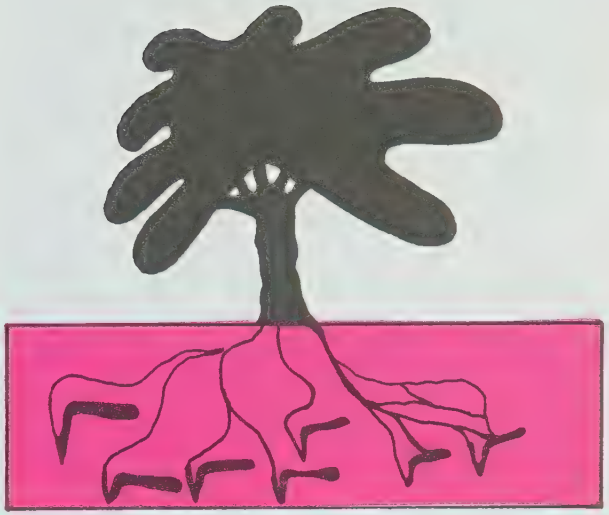
$= \frac{3\times 2\sqrt{3\times 5}}{12}$ Why?

$= \frac{\sqrt{15}}{2}$

(c) $\frac{ab^2\sqrt{a}}{\sqrt{b}} = \frac{ab^2\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$

$= \frac{ab^2\sqrt{ab}}{b}$

$= ab\sqrt{ab}$



EXERCISE 11·3

Find the value of each by using the table of squares and square roots.

1. 29^2
2. 37^2
3. 6.7^2
4. 780^2
5. 8.9^2
6. $\sqrt{14}$
7. $\sqrt{44}$
8. $\sqrt{54}$
9. $\sqrt{87}$
10. $\sqrt{94}$
11. $\sqrt{529}$
12. $\sqrt{5041}$
13. $\sqrt{7921}$
14. $\sqrt{12.96}$
15. $\sqrt{51.84}$

Rationalize the denominator and simplify in each case:

16. $\frac{\sqrt{5}}{\sqrt{2}}$
17. $\frac{\sqrt{9}}{\sqrt{2}}$
18. $\frac{\sqrt{31}}{\sqrt{2}}$
19. $\frac{\sqrt{20}}{\sqrt{3}}$
20. $\frac{\sqrt{8}}{\sqrt{5}}$
21. $2\sqrt{\frac{6}{8}}$
22. $\frac{\sqrt{x}}{\sqrt{y}}$
23. $\frac{y^2\sqrt{3}}{\sqrt{y}}$
24. $\frac{\sqrt{5}}{\sqrt{x}}$
25. $\frac{z^3\sqrt{xy}}{\sqrt{z}}$
26. $\sqrt{\frac{5}{b}}$
27. $\frac{\sqrt{m}}{\sqrt{n}}$
28. $\frac{3\sqrt{a-b}}{2\sqrt{a+b}}$
29. $\frac{\sqrt{2a+b}}{\sqrt{a-2b}}$
30. $\sqrt{\frac{3a-2b}{2a+3b}}$.

If $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, find, correct to 2 decimal places, the numerical value of each: (Rationalize the denominator before dividing.)

31. $\frac{1}{\sqrt{2}}$ 32. $\frac{3}{\sqrt{5}}$ 33. $\frac{\sqrt{2}}{\sqrt{10}}$

34. $\frac{\sqrt{6}}{\sqrt{10}}$ 35. $\frac{\sqrt{3}}{\sqrt{20}}$ 36. $\frac{2}{\sqrt{75}}$

37. $\sqrt{5} \times \sqrt{3}$ 38. $\sqrt{3} \times \sqrt{2}$

39. $\sqrt{5} \times \sqrt{4}$ 40. $\sqrt{3} \times \sqrt{6}$

41. $2\sqrt{3} \times 3\sqrt{6}$ 42. $3\sqrt{5} \times 4\sqrt{8}$.

Simplify:

43. $\sqrt{2xy} \times \sqrt{3xy}$ 44. $3\sqrt{mn} \times 2\sqrt{m^3n^3}$

45. $m\sqrt{ab} \times n\sqrt{ba}$ 46. $x\sqrt{z} \times y\sqrt{w}$

47. $2\sqrt{a-b} \times 6\sqrt{a+b}$ 48. $\frac{\sqrt{a+b} \times \sqrt{a^2-b^2}}{\sqrt{a-b}}$

49. $\frac{3\sqrt{2a-b} \times 2\sqrt{4a^2-b^2}}{\sqrt{2a+b}}$ 50. $\frac{\sqrt{xa} \times \sqrt{x^3a^3}}{\sqrt{x}}$

11·4 Using Exponents

Cartoon courtesy Better Homes and Gardens magazine. © Meredith Corporation, 1965

"Let's not get involved."

Let's get involved! Study each example. Then do as many problems in the exercise as necessary.

Example 1

Express in equivalent simplified exponential form:

$3^2 \times 3^3 \quad (-8)^2 \times (-8)^5$.

Solution

$$3^2 \times 3^3 = (3 \times 3)(3 \times 3 \times 3)$$
$$= 3^5$$

$$(-8)^2 \times (-8)^5 = (-8)(-8) \times (-8)(-8)(-8)(-8)(-8)$$
$$= (-8)^7$$

Do you know of a *short cut* method to find these answers?

I Use your method to do these.

(a) $5^4 \times 5^6$ (b) $5^{13} \times 5^{14}$

(c) $5^{\frac{1}{2}} \times 5^{\frac{3}{2}}$ (d) $6^{\frac{3}{2}} \times 6^{\frac{3}{4}}$

(e) $5^4 \times 5^{-3}$ (f) $3^2 \times 3^{-4}$

(g) $2^{\frac{4}{3}} \times 2^{-\frac{1}{3}}$ (h) $8^{-\frac{1}{4}} \times 8^{-\frac{5}{6}}$

(i) State the rule you used.

Example 2

Express each with one exponent.

$$\frac{8^5}{8^2} \qquad \frac{10^4}{10^1}$$

Solution

$$\frac{8^5}{8^2} = \frac{8 \times 8 \times 8 \times 8 \times 8}{8 \times 8} \qquad \frac{10^4}{10^1} = \frac{10 \times 10 \times 10 \times 10}{10}$$
$$= 8 \times 8 \times 8 \qquad = 10 \times 10 \times 10$$
$$= 8^3 \qquad = 10^3$$

Do you know a *short cut* to find these answers?

II Use your method to express each with one exponent.

(a) $\frac{2^3}{2^2}$ (b) $\frac{3^{16}}{3^{12}}$ (c) $\frac{3^{\frac{5}{2}}}{3^{\frac{1}{2}}}$

(d) $\frac{5^{\frac{7}{2}}}{5^3}$ (e) $\frac{4^5}{4^{-2}}$ (f) $\frac{7^6}{7^{-3}}$

(g) $\frac{9^{-6}}{9^3}$ (h) $\frac{19^{\frac{5}{2}}}{19^{\frac{3}{2}}}$

(i) State the rule you used.

(j) Evaluate: $\frac{3^4}{3^4}, \frac{a}{a}, \frac{5^7}{5^7}, \frac{8^9}{8^9}, 4^0, 3^0$.

Example 3

Express each with one exponent.

$(4^3)^2$ $(3^2)^4$

Solution

$(4^3)^2 = 4^3 \times 4^3$
$= 4^{3+3}$
$= 4^{2\times 3}$
$= 4^6$

$(3^2)^4 = 3^2 \times 3^2 \times 3^2 \times 3^2$
$= 3^{2+2+2+2}$
$= 3^{4\times 2}$
$= 3^8$

III Use your method to express each with one exponent.

(a) $(5^3)^7$ (b) $(12^4)^5$ (c) $(6^4)^{\frac{1}{2}}$

(d) $(7^{\frac{5}{2}})^{\frac{2}{3}}$ (c) $(13^{-\frac{1}{3}})^{\frac{3}{4}}$ (f) $(15^{-\frac{2}{3}})^{-\frac{3}{4}}$

(g) $(9^{-4})^{-5}$ (h) $(8^{\frac{2}{3}})^4$

(i) State the rule you used.

Example 4

Express each with a positive exponent.

14^{-3} $(\frac{3}{14})^{-\frac{3}{2}}$

Solution

$14^{-3} = (\frac{1}{14})^3$
$= \frac{1}{14^3}$

$(\frac{3}{14})^{-\frac{3}{2}} = \left(\frac{1}{\frac{3}{14}}\right)^{\frac{3}{2}}$
$= (1 \div \frac{3}{14})^{\frac{3}{2}}$
$= (\frac{14}{3})^{\frac{3}{2}}$

IV Write a rule for the type of question in Example 4, Write each with a positive exponent.

(a) 3^{-2} (b) 5^{-6} (c) $8^{-\frac{2}{3}}$

(d) $9^{-\frac{1}{4}}$ (e) $5^{-\frac{7}{6}}$ (f) $(\frac{15}{4})^{-\frac{3}{4}}$

(g) $(\frac{64}{5})^{-6}$ (h) $(\frac{40}{3})^{-\frac{2}{3}}$

(i) State the rule you used.

Observe these two rules to express a power in expanded notation.

V $(2 \times 4)^3 = 2^3 \times 4^3$

VI $(\frac{3}{4})^2 = \frac{3^2}{4^2}$

State both rules in your own words.

Express each in expanded exponential form.

(a) $(14 \times 3)^4$ (b) $(a \times b)^x$

(c) $(4a)^{\frac{3}{4}}$ (d) $(56)^{-\frac{4}{3}}$

(e) $(\frac{5}{13})^2$ (f) $\left(\frac{a}{b}\right)^m$

(g) $\left(\frac{4}{d}\right)^{\frac{n}{m}}$ (h) $\left(\frac{x}{4}\right)^{-y}$

We have used exponents which are fractions. The definition of these are given here.

$a^{\frac{1}{2}} = \sqrt{a}$ $a^{\frac{1}{3}} = \sqrt[3]{a}$ $a^{\frac{1}{x}} = \sqrt[x]{a}$ $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ or $= \sqrt[n]{a^m}$

Example 5

Evaluate each:

$9^{\frac{1}{2}}$ $27^{\frac{2}{3}}$

Solution

$9^{\frac{1}{2}} = \sqrt{9}$
$= 3$

$27^{\frac{2}{3}} = (\sqrt[3]{27})^2$
$= (3)^2$
$= 9$

VII Evaluate each:

(a) $16^{\frac{1}{2}}$ (b) $625^{\frac{1}{2}}$ (c) $49^{\frac{1}{2}}$

(d) $64^{\frac{1}{3}}$ (e) $125^{\frac{1}{3}}$ (f) $125^{\frac{2}{3}}$

(g) $64^{\frac{5}{6}}$ (h) $1000^{\frac{4}{3}}$ (i) $9^{\frac{3}{2}}$.

11·5 Exponential Growth

Have you ever wondered how a little spot of mould on a piece of bread can become so large so fast? Mould and bacteria reproduce as yeast does. If you place a growing yeast spore under a microscope and take time-lapse photographs of it you may obtain a series of pictures similar to this set.

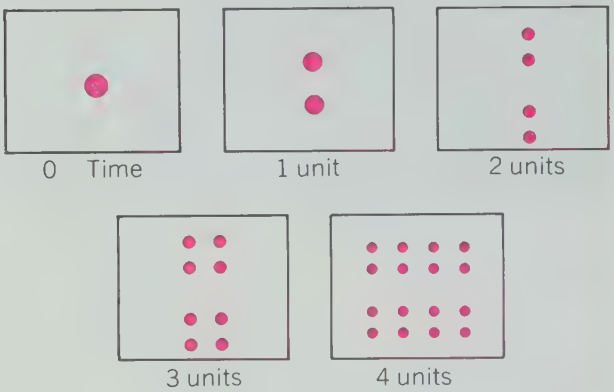
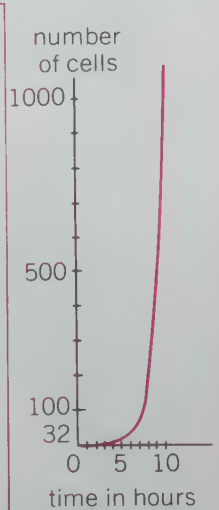


Figure 11-1

Each cell divides into two cells in one unit of time. The two cells divide into four cells in a second unit of time, the four into eight, the eight into sixteen and so on. For ease of discussion we shall call this unit of time 1 hour. We can set up this table. We show the graph also.

Number of cells of yeast

Time (hour)	*Standard numerals*	*Exponential notation*
0	1	2^0
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
.	.	.
.	.	.
.	.	.
10	1024	2^{10}
.	.	.
.	.	.
.	.	.
n		2^n



Growth of this type is called *exponential growth*. It is common at some stages of growth in many living organisms. There are factors which affect the exponential growth: factors such as food supply, waste removal, growing environment (heat, light, etc.). Normally, there is what scientists call a *lag* phase at the early stages of the growth period, followed by an *exponential* phase and then by an *approach to stationary* phase. These phases may be observed in Project 1.

Project

Project 1

Purpose

To experience and record the growth of mould.

Method

General Outline: A clear gelatin (growth medium) will be put into pie plates which have graph paper glued on the bottom. Mould spores from the air will contaminate the growth medium. This culture will be allowed to grow for two weeks in a warm, dark place. On the third, and each day thereafter, the area the culture covers will be recorded.

Specific Steps:

1. Obtain the following materials

pie plates –	aluminum foil or paper – about 6 to 8 inches in diameter – 1 per team.
graph paper –	10 lines per inch – 1 sheet per team.
–	large sheets (20″ × 24″) of 1-inch squared paper for wall graph – 3 sheets per team (or 3 sheets per student of individual graph paper – 10 lines per inch).
Saran Wrap –	to cover pie plates
masking tape –	to tape Saran Wrap over pie plates
Oxo cubes –	1 cube per class
gelatin –	colorless if possible – 1 package per class. (1 package of clear gelatin makes a pint of solution.)

2. Cut graph paper to fit bottom of pie plate and glue into place. (While balance of class is preparing the paper plates, two students should prepare the growth medium.)

3. Prepare the growth medium.

   Dissolve Oxo cube in 1 cup of warm water.

   Dissolve gelatin-mix in 1 cup of water.

   Combine the two solutions.

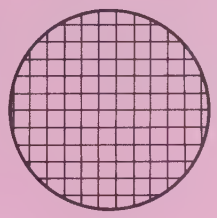


Diagram A

4. Pour enough solution into each plate so that the graph paper is just covered.

5. Allow gelatin to *set* (as it cools it *sets*). Since it is open to the air, mould spores will settle from the air onto the growth medium.

6. When set, cover with Saran Wrap and tape down.

7. Place in a warm, dark place.

8. Observe daily and record your observations.

9. To record; DO NOT REMOVE THE SARAN WRAP.

   Each small square will be the unit of area used. It will be necessary to estimate the area that the culture (mould) covers. A number of small dots make up the equivalent of one small square.

   In diagram B one could estimate that four dots equal one small square. As the dots grow in size each will soon cover a small square. Record the area (in small squares).

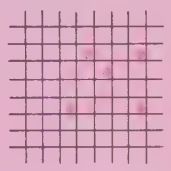


Diagram B

10. From the first day that the mould appears (about day 3) record daily the area covered by the mould.

    Record the data in a table as shown in the table below.

Each team records the data in columns A, B, C, in three separate wall graphs. (Or each student records on small individual sheets the data from columns A, B and C.)

Table 1

	A		B		C	
Day	Total area (in small squares)		Increase in area over previous day		Percent increase $= \frac{\text{increase in area}}{\text{total area of previous day}} \times 100$	
1 Monday 2 Tuesday ⋮ 5 Friday 6 Saturday 7 Sunday 8 Monday ⋮ 13 Saturday 14 Sunday.	extrapolated value	interpolated value	extrapolated value	interpolated value	extrapolated value	interpolated value

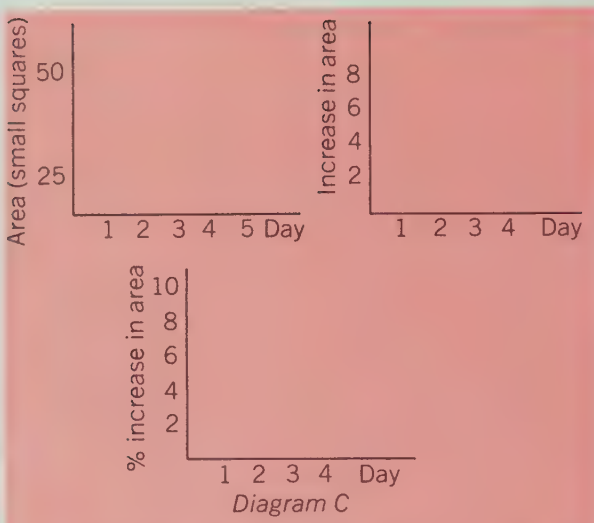


Diagram C

11. After three days recording mark in red on each graph your prediction of the next day's data. This is informal extrapolation. Show the actual data in blue.

12. Each weekend you will not be at school to record the data. On Friday, extrapolate to obtain an estimate of the growth for these two days. On Monday, after recording the data, interpolate for Saturday and Sunday.
    (a) Do the extrapolated values and interpolated values coincide? Which do you think is nearer the correct value? Why?
    (b) In which cases and in which way is your interpolated value better than the extrapolated value?

13. Does the growth curve (Graph A) show a *lag* phase, an *exponential* phase and an *approach to stationary* phase? A lag phase is the slow start in the growth pattern. The exponential phase is the period of rapid growth. The approach to stationary phase is the slowing-down stage in growth pattern. Which days make up the
    (a) lag phase?
    (b) the exponential phase?
    (c) the approach to stationary phase?

14. When the project is completed dispose of the paper plates by placing them in a paper bag without removing the Saran Wrap. Then place in the garbage disposal.

EXERCISE 11·5

1. Refer to Table 1. Draw a graph showing the relation between time and number of cells for $0 \leq t \leq 9$.

2. The following chart shows the growth pattern of a bamboo – which has a very rapid growth.

age (in weeks)	*height (in metres)*	*weight (in grams)*
1	0.5	10
2	1.7	30
3	3.9	80
4	6.0	140
5	8.5	200
6	10.6	310
7	12.4	425
8	13.8	550
9	14.2	680
10	14.5	720
11	14.7	780
12	14.8	794
13	14.8	800
14	14.8	804
15	14.8	807

Refer to Table 1 in Project 1. Complete a table like Table 1 with column B and column C for each of the height and the weight data. Then plot graphs for each set of data as you did in Project 1.

Describe in your own words what you observed in the growth patterns of bamboo. Relate height to weight. Did the weight continue to grow after the increase in height ceased? Explain why.

Bamboo in the Chao Phraya River, Thailand.

11·6 Graphing Growth Patterns

Let us consider three quiz programs in which the prize money varies with the number of questions answered correctly. The rules of each program are set out below.

Program A

The contestant gets $2 for each correct answer.

Program B

The contestant receives $1 for the first correct answer, $3 more for the second correct answer, $5 more for the third correct answer, $7 more for the fourth correct answer, and so on.

Program C

The contestant receives $2 for the first correct answer, $4 for two correct answers, $8 for three correct answers, and so on.

In which program would you prefer to take part? Can you explain your choice?

Let us show *growth* of the prize money as the number of questions answered correctly increases to 10. In your exercise book set up a table as shown for each program. Complete the tables.

x = number of questions answered correctly

y = accumulated amount of money earned

Program A

x	1	2	3	4	5	6	7	8	9	10
y	2	4	6							

Program B

x	1	2	3	4	5	6	7	8	9	10
y	1	4	9	16						

Program C

x	1	2	3	4	5	6	7	8	9	10
y	2	4	8							

From the tables it becomes clearly evident which program pays out the most in prize money if each contestant answers 10 questions correctly. An analysis of the three tables tells us much about the three quiz programs and the amounts one would win depending on the number of questions correctly answered. It may be easier to compare the three sets of data in graph form.

On the same set of axes draw the graphs to represent the three situations. While these should properly be shown by dot graphs, the *story* may be more clearly interpreted from a continuous line graph. Draw a smooth curve through each set of points to obtain three different curves. Draw large graphs. Then answer the questions in the exercise.

EXERCISE 11·6

1. In what situation would it have been best for a participant to have been on Program C?

2. In what situation would it have been best for a participant to have been on Program B?

3. Are there any circumstances under which Program A would be the best?

4. Describe in your own words the *growth* of the prize money in each program. You may wish to refer to *the slope* of the growth curve which describes the particular program prize money.

5. Each of the curves in the graph is described by one of the following sets. Select the set that describes each curve.

   $I = \{(x,y) | y = 2^x, x, y \in R^+\}$

   $II = \{(x,y) | y = x^2, x, y \in R^+\}$

   $III = \{(x,y) | y = 3x - 2, x, y \in R^+\}$

   $IV = \{(x,y) | y = 2x, x, y \in R^+\}$

6. Why should the graphs of the earnings on the three different quiz programs properly be shown as *dot* graphs instead of continuous curve graphs?

7. At one time there was a quiz program (like Program C) in which the contestant received \$2 for answering correctly the first question, or he could try for \$4 by answering a second question. When the contestant reached the \$1,024 question the amount was rounded to \$1,000 and the quiz program continued. The maximum the contestant could win was \$64,000. If the contestant failed to answer a question correctly he received no money.

   (a) How many questions need a contestant answer successfully in order to win \$64,000?

   (b) How many questions need a contestant answer successfully in order to win half of the \$64,000?

   (c) What was the average amount he would receive per question if he received \$64,000?

   (d) One contestant quit after answering correctly ten questions. What was the average amount he received per question?

   (e) If the contestant could have continued until he reached \$1,000,000 how many more questions beyond the \$64,000 question would he have had to answer?

   (f) On the average what was the amount he would receive per question answered if he received \$1,000,000?

   (g) Can you account for the difference in the two answers to questions (d) and (f)?

8. Set up a table and draw a graph to show how the *area* of a square *grows* as the length of a side *grows*. *Hint!* For ease, start with a square with length of 1 unit. Then let it *grow* to a square with length of 2 units, and so on.
   Write the relation in standard set builder notation.

9. Set up a table and draw a graph, with respect to the same set of axes as question 8, to show how the volume of a cube *grows* as the length of a side *grows*. *Hint!* Reread the hint in question 8. Write the relation in standard set builder notation.

10. Describe the growth rate of the area of a square versus the volume of a cube as the measure of a side of each increases.

11·7 Exponential Decay

We call the expression a^x an *exponential expression*. We call the relation $y = a^x$ an *exponential relation*. If we examine the graphs of $y = 2^x$, $x \in R^+$ (previous section) we find that for each element of the domain there is one and only one corresponding element in the range. Therefore, the relation $y = 2^x$, $x \in R^+$, is a function—an *exponential function*. In general, all relations of the form $\{(x,y) | y = a^x, x \in R^+\}$ are exponential functions.

The previous sections pointed out that not all growth rates are exponential. In the work that follows considerable space will be given to exponential functions.

Before continuing with exponential growth, we shall introduce here *exponential decay*. As you no doubt guessed, the term exponential decay refers to the decrease in size, number, or quantity. A very simple example is portrayed in an old philosophical puzzle.

A frog is 10 feet from the edge of a pond. It decides to jump half the distance to the pond in the first jump, then half the remaining distance in each succeeding jump. How many jumps does it take to get to the pond's edge?

When we try to solve this problem we come up with a surprising answer. Let us set up a table of values.

n = number of jumps	0	1	2	3	4	5	6
d = distance remaining (ft)	10	5	2.5	1.25	.625	3125	

Now we will show this on a graph:

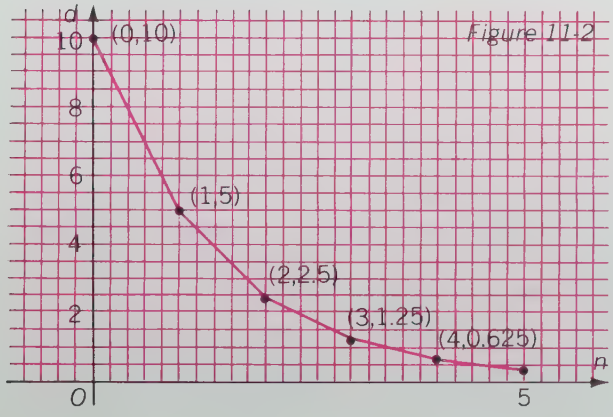


Figure 11-2

From the table and from the graph you can see that after each jump there is always going to be a certain distance – small though it may be – left. Theoretically, the frog never gets to the pond. *But* we also know that for all *practical purposes* it does reach the pond.

The problems we have discussed in both exponential growth and decay have involved nonnegative values (positive numbers or zero) for the domain. Can we involve negative values for the domain?

In the problem just discussed the jumps after the 10-foot mark were positive numbers. Then it is reasonable to assign negative values to those jumps before the 10-foot mark. The table would look like this.

n	−3	−2	−1	0	1	2
d				10	5	2.5

Now if the frog had been following the same rule in all of his jumps then at the jump numbered −1 he would have been 20 feet from the pond, and at jump numbered −2, he would have been 40 feet from the pond. Hence, the table partly completed now looks like this.

n	−4	−3	−2	−1	0	1	2	3
d		80	40	20	10	5	2.5	

Graphically the information is depicted in Figure 11-3.

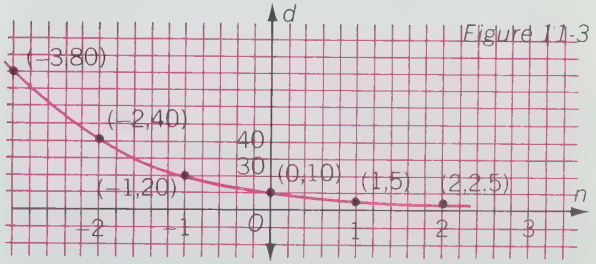


Figure 11-3

To represent this relation in equation form requires a different approach. Let d = distance left to jump and n the number of the jump.

Then after 1 jump $d = 10(\frac{1}{2})$,
after 2 jumps $d = 10(\frac{1}{2})(\frac{1}{2}) = 10(\frac{1}{2})^2$,
after 3 jumps $d = 10(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = 10(\frac{1}{2})^3$,
and after n jumps $d = 10\underbrace{(\frac{1}{2})(\frac{1}{2})\ldots(\frac{1}{2})}_{n\ \frac{1}{2}\text{'s}} = 10(\frac{1}{2})^n$.

Therefore, the relation is $d = 10(\frac{1}{2})^n$ or $d = 10(2)^{-n}$.

In general, the exponential decay relation is $d = a^{-n} (a > 1)$. In this general case the constant 10 is generalized to 1 unit.

From the graph we see that this is a function for all real values of n. Generalizing from this one example, we say

> The relation $\{(x,y) | y = a^x, x \in R, a \neq 0\}$ is an exponential function.

An example of exponential decay discussed frequently today is radioactive decay. In the process of disintegration, radioactive material emits particles and loses mass. Consider the case of polonium. If we start with 1 gram of polonium, after 140 days there will be only $\frac{1}{2}$ gram of polonium left. This period of 140 days is called the *half-life* of polonium.

Geiger counter (above) detects amount of radiation from radioactive elements. Table (below) shows half-lives of several elements.

Element	*Atomic number*	*Atomic mass*	*Particle given off*	*Half-life*
Uranium	92	238	Alpha	4,500 million years
Thorium	90	234	Beta	24 days
Protoactinium	91	234	Beta	1.1 minutes
Uranium	92	234	Alpha	250,000 years
Thorium	90	230	Alpha	80,000 years
Radium	88	226	Alpha	1,600 years
Radon	86	222	Alpha	3.8 days
Polonium	84	218	Alpha	3 minutes
Lead	82	214	Beta	27 minutes
Bismuth	83	214	Alpha or Beta	20 minutes
Polonium	84	214	Alpha	0.001 second
Thallium	81	210	Beta	1.3 minutes
Lead	82	210	Beta	20 years
Bismuth	83	210	Beta	5 days
Polonium	84	210	Alpha	140 days
Lead	82	206	——	Stable

After another 140 days there will be only half of the $\frac{1}{2}$ gram (or $\frac{1}{4}$ gram) left. After the third period of 140 days there will be only $\frac{1}{8}$ gram left. As you can see, this situation is analogous to the *frog* situation.

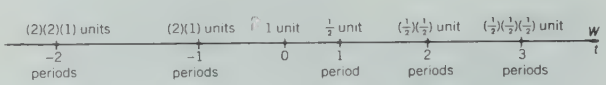
The number of units left after 1 period is $\frac{1}{2}$.

The number of units left after 2 periods is $(\frac{1}{2})^2$.

The number of units left after 3 periods is $(\frac{1}{2})^3$.

The number of units left after t periods is $(\frac{1}{2})^t$.

If W represents the number of grams of polonium left and t represents the number of periods of 140 days, then, $W = (\frac{1}{2})^t$ or $W = 2^{-t}$ represents the exponential decay function.



Graphically this is shown in Figure 11-4.

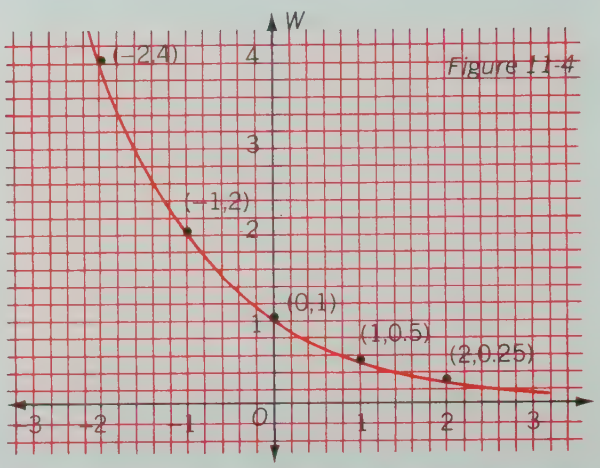


Figure 11-4

Different radioactive materials disintegrate at different rates. The time for a given mass of material to disintegrate to one-half that mass is called its half-life. A large amount of radioactive material loses more grams of weight per period of time than a small piece of the same material.

Half-lives of radioactive substances vary from extremely short periods of time (microseconds) to many billions of years.

EXERCISE 11·7

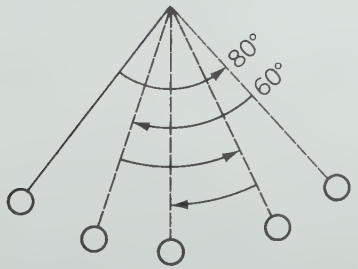
1. Look at the graph in Figure 11-3. n represents the number of jumps after the 10-foot position. What does an n value of -2 represent?

2. Look at the graph in Figure 11-4. What is the approximate amount of polonium left at $t = \frac{1}{2}$? $t = 1\frac{1}{4}$? $t = 0.7$? $t = \sqrt{3}$?

3. Explain in your own words what is meant in question 2 when it asks the approximate amount of polonium left at $t = \frac{1}{2}$? and $t = \sqrt{3}$?

4. A certain isotope of mercury has a half-life of approximately one day. An experiment with 800 grams is conducted.
   (a) Make a table to show the weight of matter left after 1, 2, 3, 4, 5, and 6 days.
   (b) Extend the table to show the weight of matter present 1, 2, and 3 days before the experiment.
   (c) Record on a graph the data from (a) and (b).
   (d) How many days after the experiment began will there be less than 1 gram left?

5. A pendulum is set into motion. After a few oscillations the degree measure of a swing is 80. It decays $\frac{3}{4}$ of its swing each time.



   (a) Set up a table to show the degree measure of swing numbers 1, 2, 3, 4, 5. (Round the degree measure to the nearest unit.)
   (b) If the pendulum made three swings before it was measured, calculate the measures of its first three swings.
   (c) Show the measures of the swings of (a) and (b) on a graph.
   (d) Should this be a dot graph or a line graph? Why?
   (e) Write the exponential function of the pendulum in set builder notation.

6. (a) A ball is dropped from a height of 12 feet. On each bounce it rises $\frac{2}{3}$ the height of the previous bounce. Set up a table to show the height of 5 successive bounces. Then record the data in a graph.
   (b) Show on the same graph the heights of the bounces if the ball rises only $\frac{1}{3}$ the height of the previous bounce.
   (c) Is the second graph just one-half that of the previous graph?

Toronto: Growth Rate 37%

If the population of Toronto is 2.7 million, what will it be in one year? two years? five years? This is exponential growth.

11·8 Exponential Functions

We have assigned a meaning to the symbol 10^x for each rational value of x. For instance, $10^{\frac{1}{2}} = \sqrt{10}$, $10^{\frac{3}{2}} = \sqrt{10^3}$, $10^0 = 1$, $10^{-\frac{1}{2}} = \frac{1}{\sqrt{10}}$ and $10^{1.6} = 10^{\frac{16}{10}} = \sqrt[10]{10^{16}}$. We can extend this meaning to include all real values of x; that means the expression $10^{\sqrt{2}}$ has a meaning also.

We shall now construct a table of values for

$$\{(x,y) | y = 10^x, x \in R\}.$$

x	−2	−1	0	.25	.50	.75	1	1.5	
y									

Now, find the value of y when $x = 0$.

$$y = 10^0$$
$$y = 1.$$

Draw the table in your book and fill in the values of y as we find them.

Find the value of y when $x = 1$.

$$y = 10^1$$
$$y = 10$$

To find the value of y when $x = 0.5$ we have

$$y = 10^{.5}$$
$$= 10^{\frac{1}{2}}$$
$$= \sqrt{10}$$

By Newton's method, or by tables or by slide rule, we find

$$\sqrt{10} \simeq 3.2$$
$$\therefore \quad y \simeq 3.2$$

To find the value of y when $x = 0.25$ we use the fact that $0.25 = \frac{1}{4}$.

$$y = 10^{\frac{1}{4}}$$
$$= (10^{\frac{1}{2}})^{\frac{1}{2}}$$
$$= (3.2)^{\frac{1}{2}}$$
$$= \sqrt{3.2}$$

Then using tables we get

$$y \simeq 1.8.$$

Can you find the value of y when $x = 0.75$?

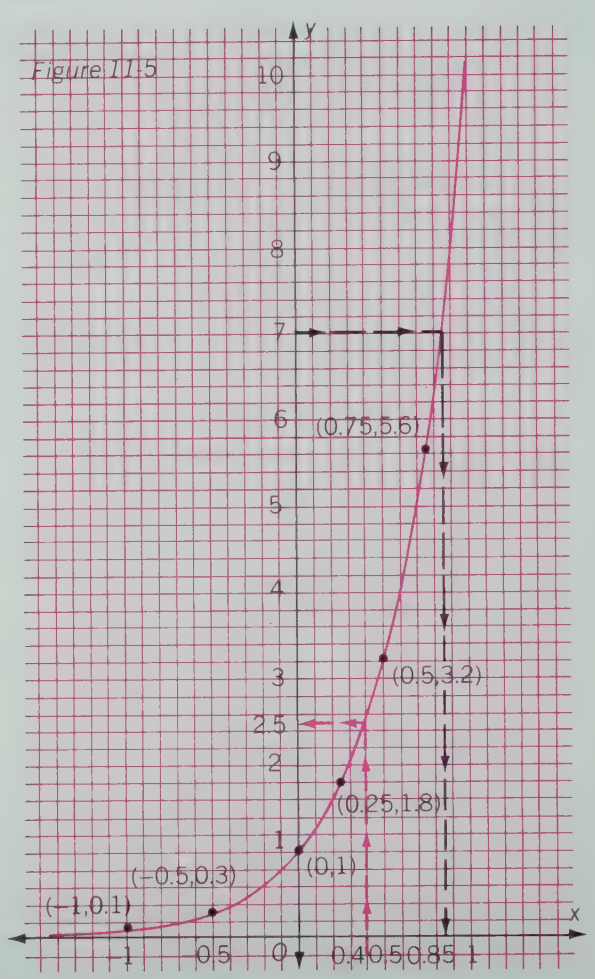
Hint! Use the fact that $\frac{3}{4} = \frac{1}{4} \times 3$.

Fill in the balance of the table.

You should have the following table.

x	−1	−0.5	0	0.25	0.5	0.75	1		
y	0.1	0.3	1	1.8	3.2	5.6	10		

Figure 11-5 shows this information in graph form. A continuous curve has been drawn to contain the points.



From this graph we are able to approximate more values for y. Use the graph to find approximate values of each. The first is done for you (see colored line on graph).

$10^{0.4} \simeq 2.5$ $\quad 10^{0.6} \simeq \blacksquare$ $\quad 10^{-0.3} \simeq \blacksquare$ $\quad 10^{-0.8} \simeq \blacksquare$

From the graph we are also able to get the information that will allow us to express a number as a power of 10. For example, if we wish to write the number 7 as a power of 10 we write

$$7 = 10^x.$$

Since $y = 7$, we want to find x.

Look at the graph in Figure 11-5. We locate 7 on the y-axis then follow the dotted line to the curve, then to the x-axis. Here we find $x \simeq 0.85$. Hence,

$$7 \simeq 10^{0.85}$$

Use the graph to express as a power of 10 each number.

$8.5 = \blacksquare$ $\quad 4 = \blacksquare$ $\quad 1.5 = \blacksquare$ $\quad 0.5 = \blacksquare$

Example 1

Multiply, using the graph in Figure 11-5:

$$4.5 \times 1.6$$

Solution

$4.5 \times 1.6 \simeq 10^{.65} \times 10^{.2}$ $\quad \because 4.5 \simeq 10^{.65}$ and $1.6 \simeq 10^{.2}$ from graph

$= 10^{.65 + .2}$ $\quad$ Law of Products

$= 10^{.85}$

$\simeq 7$ $\quad$ From graph $10^{.85} \simeq 7$

Example 2

Divide, using the graph in Figure 11-5:

$$8.1 \div 2.6.$$

Solution

$$8.1 \div 2.6 \simeq 10^{.9} \div 10^{.4}$$
$$= 10^{.9 - .4}$$
$$= 10^{.5}$$
$$\simeq 3$$

Check this work by arithmetic.

Example 3

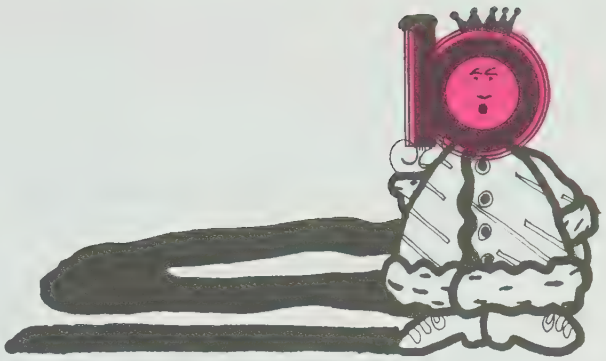
Evaluate $(7.5)^{\frac{2}{3}}$ using the graph in Figure 11-5.

Solution

$(7.5)^{\frac{2}{3}} \simeq (10^{.87})^{\frac{2}{3}}$ $\quad \because 7.5 \simeq 10^{.87}$ from graph

$= 10^{.58}$

$\simeq 3.8$ $\quad \because 10^{.58} \simeq 3.8$ from graph



What is meant by a power of ten?

EXERCISE 11·8

Use the graph in Figure 11-5 to do the following questions.

1. Express as a power of 10.
   (a) 1.2 $\quad$ (b) 4.6 $\quad$ (c) 7.8 $\quad$ (d) 8.5
2. Express each in decimal notation.
   (a) $10^{.35}$ $\quad$ (b) $10^{.25}$ $\quad$ (c) $10^{.75}$ $\quad$ (d) $10^{.67}$
3. Find the product in each case using Figure 11-5.
   (a) 4.2×2.3 $\quad$ (b) 5.6×1.6
   (c) 1.7×3.6 $\quad$ (d) 3.1×3.1
4. Find the quotient using Figure 11-5.
   (a) $9.6 \div 3.2$ $\quad$ (b) $8.4 \div 2.5$
   (c) $7.6 \div 6.5$ $\quad$ (d) $9.2 \div 2.6$
5. Express in decimal notation.
   (a) $(3.5)^{1.5}$ $\quad$ (b) $(4.5)^{\frac{1}{3}}$ $\quad$ (c) $(7.8)^{\frac{3}{4}}$ $\quad$ (d) $(8)^{\frac{2}{3}}$
6. Use Figure 11-5 to evaluate each of the following.
   (a) $10^{\frac{1}{4}}$ $\quad$ (b) $10^{\frac{3}{4}}$
   (c) $10^{.28}$ $\quad$ (d) $3.8 \times 6.3 \times 5.6$
   (e) $3.4 \div 6.5$ $\quad$ (f) $4.5 \div 9$
   (g) $(2.5)^{.25}$ $\quad$ (h) $(8.6)^{1.2}$
   (i) $\dfrac{3.4 \times 5.6}{3.8}$ $\quad$ (j) $\dfrac{4.8 \times 6.8 \times 3.2}{9.2 \times 8.4}$

11·9 More About Exponents and Calculations

We shall review *standard notation* or *scientific notation* in order that we might use it in subsequent work.

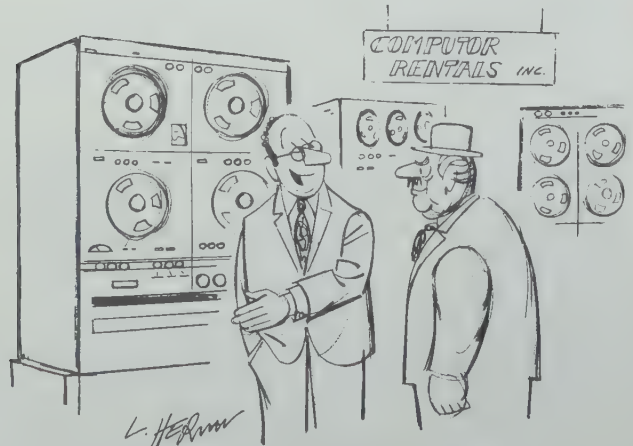
$$\begin{aligned} 1453 &= 1.453 \times 10^3 \\ 276 &= 2.76 \times 10^2 \\ 53 &= 5.3 \times 10 \\ 4 &= 4 \times 10^0 \quad \text{(usually just '4')} \\ 0.6 &= 6 \times 10^{-1} \\ 0.07 &= 7 \times 10^{-2} \\ 0.009 &= 9 \times 10^{-3} \end{aligned}$$

The above examples illustrate how we can express a number as the product of a number equal to or greater than 1 but less than 10, and a power of 10. If the number $a \times 10^b$ is in standard notation then

$$1 \leq a < 10 \text{ and } b \in I.$$

Before continuing with this section, turn to the exercise and do as many parts of questions 1 and 2 as necessary to become adept at writing in standard notation.

Now we shall use the graph of Figure 11-5 on page 206 to do computations involving large and small numbers.



"This model will do the work of four conscientious employees or nineteen relatives."

Example 1

Evaluate 3760×0.16 using the graph in Figure 11-5.

Solution

First, we write each number in standard notation.

$$\begin{aligned} 3760 &= 3.76 \times 10^3 \\ 0.16 &= 1.6 \times 10^{-1} \\ \therefore \quad 3760 \times 0.16 &= (3.76 \times 10^3) \times (1.6 \times 10^{-1}) \\ &= 3.76 \times 1.6 \times 10^3 \times 10^{-1} \quad \text{Assoc. and Comm. Property} \end{aligned}$$

Next, express 3.76 and 1.6 in exponential form. See Figure 11-5.

$$\begin{aligned} 3.76 &\simeq 10^{.58} \\ 1.6 &\simeq 10^{.2} \\ 3760 \times 0.16 &\simeq 10^{.58} \times 10^3 \times 10^{.2} \times 10^{-1} \\ &= 10^{3.58} \times 10^{-.8} \\ &= 10^{2.78} \\ &= 10^{.78} \times 10^2 \quad \left\{ \text{Notice we write this in standard notation.} \right. \\ &\simeq 6 \times 10^2 \\ &= 600 \end{aligned}$$

We obtain the answer of 600. Use arithmetic to check the accuracy of this answer.

Example 2

Evaluate $0.324 \div 87$ using the graph in Figure 11-5.

Solution

Follow the steps established in Example 1.

$$\begin{aligned} 0.324 &= 3.24 \times 10^{-1} \\ 87 &= 8.7 \times 10^1 \\ 0.324 \div 87 &= (3.24 \times 10^{-1}) \div (8.7 \times 10^1) \\ &\simeq (10^{.5} \times 10^{-1}) \div (10^{.94} \times 10^1) \\ &= 10^{-.5} \div 10^{1.94} \\ &= 10^{-.5-1.94} \quad \text{Why?} \\ (1) \qquad &= 10^{-2.44} \\ (2) \qquad &= 10^{.56} \times 10^{-3} \quad \text{In standard notation} \\ &\simeq 3.6 \times 10^{-3} \quad \text{From the graph} \\ &= 0.0036 \end{aligned}$$

Mult. and Div. of powers of the same base

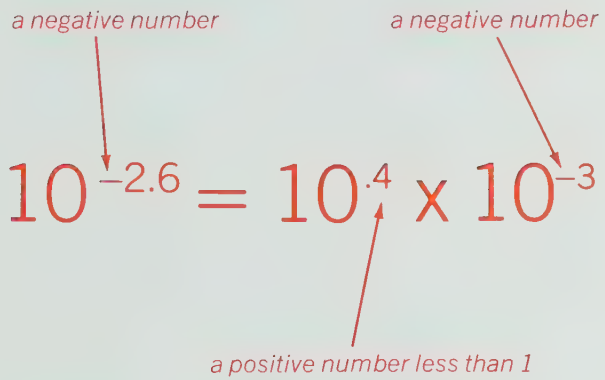
$$0.324 \div 87 \simeq 0.0036$$

You will have noticed one *tricky* step. It is the step from (1) to (2). $10^{-2.44} = 10^{.56} \times 10^{-3}$. This step is fairly obvious when written

$$10^{.56} \times 10^{-3} = 10^{.56-3}$$
$$= 10^{-2.44}$$

We should look at several of these.

$$10^{-3.2} = 10^{.8} \times 10^{-4}$$
$$10^{-1.8} = 10^{.2} \times 10^{-2}$$



Example 3

Write 0.0634 in exponential form.

Solution

$0.0634 = 6.34 \times 10^{-2}$
$= 10^{.8} \times 10^{-2}$ $\quad \because 6.34 = 10^{.8}$ from graph.
$= 10^{-1.2}$

Now do questions 3, 4, and 5 in Exercise 11·9 before continuing this section.

Example 4

Evaluate $\sqrt[3]{66}$ using the graph in Figure 11-5.

Solution

$\sqrt[3]{66} = 66^{\frac{1}{3}}$
$= (6.6 \times 10^{1})^{\frac{1}{3}}$
$\simeq (10^{.82} \times 10^{1})^{\frac{1}{3}}$ $\quad \because 6.6 \simeq 10^{.82}$ from graph.
$= (10^{1.82})^{\frac{1}{3}}$
$\simeq 10^{.61}$ $\quad$ Power Law
$\simeq 4.1$ $\quad \because 10^{.61} \simeq 4.1$ from graph.

EXERCISE 11·9

1. Write each in standard notation.

(a) 356	(b) 47645	(c) 76.41
(d) 0.365	(e) 0.0567	(f) 8.63
(g) 14.03	(h) 0.00007	(i) 876543.1
(j) 46.11	(k) 0.032	(l) 0.0076

2. Write each in decimal notation.

(a) 3.41×10^{2}	(b) 8.62×10^{-2}
(c) 1.86×10^{5}	(d) 8.8×10^{1}
(e) 9.6×10^{7}	(f) 4.1×10^{-6}
(g) 7.632×10^{3}	(h) 3.4567×10^{-2}
(i) 7.61×10^{3}	(j) 3.33×10^{4}
(k) 1.2×10^{-4}	(l) 9.6×10^{0}

3. Write each in the form 10^{x}.

(a) $10^{.4} \times 10^{-2}$	(b) $10^{.6} \times 10^{-1}$
(c) $10^{.33} \times 10^{-3}$	(d) $10^{.23} \times 10^{-2}$
(e) $10^{.54} \times 10^{-4}$	(f) $10^{.45} \times 10^{-6}$
(g) $10^{.91} \times 10^{-1}$	(h) $10^{.78} \times 10^{-2}$

4. Write each number as the product of two powers of 10—one with a positive exponent and the other with a negative exponent.

(a) $10^{-1.2}$	(b) $10^{-4.2}$	(c) $10^{-2.6}$
(d) $10^{-.4}$	(e) $10^{-3.2}$	(f) $10^{-.6}$
(g) $10^{-2.2}$	(h) $10^{-1.7}$	(i) $10^{-2.32}$

5. Write each number in question 1 in exponential form.

6. Use the graph and the method illustrated in this section to evaluate each.

(a) 37×420	(b) 24×97
(c) 1.4×0.74	(d) $.86 \times .036$
(e) $456 \times .0026$	(f) $98 \times .87$
(g) $71 \div 3.6$	(h) $64 \div .42$
(i) $35 \div 76$	(j) $2.67 \div 64$
(k) $.006 \div 3.6$	(l) $4370 \div 8670$
(m) $\sqrt{80}$	(n) $\sqrt{1760}$

11·10 Use of Tables of Exponents With Base Ten

We have been working with exponential expressions such as $10^{1.2}$, $10^{-1.8}$, and $10^{2.4}$. In general, we have used 10^x, $x \in R$. The general form of the exponential function we have used extensively in the last two sections is $y = 10^x$, $x \in R$. Since the accuracy of the work we have done by using the graph of Figure 11-5 is limited, a table has been developed which allows greater accuracy. The table for $y = 10^x$ is given in the appendix. A portion of the table is given in Table 2.

Table 2

Table of $y = 10^x$

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	3186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

To demonstrate the use of the table the following examples are presented.

Example 1

Write each number in exponential form to base 10.

(a) 2.95 (b) 0.0586 (c) 8230

Solution

(a) Starting with $y = 10^x$

$2.95 = 10^x$

We look for 295 in the table. We go down the y column to 29 then we go over to the column headed 5. Here we see the number 4698 in color. This number, 4698, with a decimal in front of it, is the exponent. Therefore,

$$2.95 \simeq 10^{.4698}$$

(b) First express the number in standard notation.

(1) $0.0586 \simeq 5.86 \times 10^{-2}$

Now working with just 5.86.

$5.86 = 10^x$.

Looking down the y column to 58, then in the same row across to the column headed 6. Here we see the number 7679. Therefore,

$5.86 \simeq 10^{.7679}$

(2) $0.0586 = 5.86 \times 10^{-2}$

$\simeq 10^{.7679} \times 10^{-2}$

$= 10^{-2+.7679}$

We will find it most convenient to leave it in the form $10^{-2+.7679}$. The reason will be obvious when you work with logarithms later.

(c) Express 8230 in standard notation.

$8230 = 8.230 \times 10^3$

In the table $8.230 = 10^{.9154}$

$8230 \simeq 10^{.9154} \times 10^3$

$= 10^{3+.9154}$.

Example 2

Write each of the following in decimal notation.

(a) $10^{3+.1553}$ (b) $10^{-2+.7657}$

(c) $10^{-1.0825}$

Solution

(a) The table is for $y = 10^x$. We must reduce the question to this form.

$$y = 10^{3+.1553}$$
$$= 10^{.1553} \times 10^3$$

We can evaluate $10^{.1553}$ from the table.

$$x = 0.1553$$

We find in the body of the table the number 1553 (notice, no decimal now). The y-value for this is 143 and for 0.1553, $y = 1.43$. Now we know

$$y = 10^{.1553} \times 10^3$$
$$\simeq 1.43 \times 10^3$$
$$= 1430.$$

$$\therefore \quad 10^{3+.1553} \simeq 1430.$$

(b) $10^{-2+.7657} \times 10^{-2}$

From the table we find $10^{.7657} \simeq 5.83$ by locating 7657 in the table.

$$\therefore \quad 10^{.7657} \times 10^{-2} \simeq 5.83 \times 10^{-2}$$
$$= 0.0583$$

(c) $10^{-1.0825}$ is written in $10^{.9175} \times 10^{-2}$.
From the table we find $10^{.9175} \simeq 8.27$.

$$\therefore \quad 10^{.9175} \times 10^{-2} \simeq 8.27 \times 10^{-2}$$
$$= 0.0827$$

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
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17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962

EXERCISE 11·10

Use the table of $y = 10^x$ on page 265.

1. Write each in exponential form to base 10 as indicated in this section.

   (a) 3.78 (b) 4.68 (c) 53.6

   (d) 7680 (e) .00765 (f) .415

   (g) .0761 (h) .000316 (i) .00111

2. Write each in decimal notation.

   (a) $10^{1.6385}$ (b) $10^{2.7543}$

   (c) $10^{3.8432}$ (d) $10^{-1+.2253}$

   (e) $10^{-1.5935}$ (f) $10^{-2.5003}$

   (g) $10^{-1+.4200}$ (h) $10^{-4+.7110}$

   (i) $10^{-3.2000}$ (j) $10^{6.7364}$

Use Table 2 to evaluate each of the following. Use the method described in Section 11·5.

3. (a) 476×3.76 (b) 1340×97.6

   (c) 417×76.3 (d) $876 \div 28.6$

   (e) $961 \div 362$ (f) $231 \div 49.6$

   (g) $4.14 \div 987$ (h) $\sqrt[4]{124}$

   (i) $\sqrt[3]{88^5}$ (j) $\sqrt[5]{125^3}$

   (k) $\sqrt{84} \times \sqrt{76}$ (l) $\dfrac{934 \times 761}{423}$

11·11 Logarithms

Another name for the exponent of a power is *logarithm*. Since it is possible to have different bases in a power, we must name the base when we talk about logarithms. In Table 2, Section 11·6, the base is 10. This is the base in the common logarithms. We shall now compare the exponential function and the logarithmic function.

exponential function	logarithmic function
$100 = 10^2$	$\text{logarithm}_{10}100 = 2$
$1000 = 10^3$	$\text{logarithm}_{10}1000 = 3$
$5.5 = 10^{.7404}$	$\text{logarithm}_{10}5.5 = .7404$

The abbreviation for logarithm is *log*.

$5.5 = 10^{.7404}$ is read: "The number 5.5 equals the base 10 to the exponent .7404."

$\log_{10}5.5 = .7404$ is read: "Logarithm of the number 5.5 to the base 10 is .7404."

Compare the two statements and note the relation between exponential functions and logarithmic functions.

Since Table 2 gives the logarithms of numbers it is called a *Table of Logarithms*.

Example 1

Express in logarithmic form:

$$74.4 = 10^{1.8716}$$

Solution

The logarithmic form is

$$\log_{10}74.4 = 1.8716.$$

Example 2

Express in exponential form:

$$\log_{10}886 = 2.9474.$$

Solution

The exponential form is

$$886 = 10^{2.9474}.$$

Example 3

Express 143 in logarithmic form.

Solution

Express 143 in standard form.

$$143 = 1.43 \times 10^2$$

Now express in exponential form.

$$\begin{aligned} 143 &= 10^{.1553} \times 10^2 \\ &= 10^{2.1553} \end{aligned}$$

Now express in logarithmic form.

$$\therefore \quad \log_{10}143 = 2.1553$$

Example 4

Each of the following is a logarithm (to the base 10) of a certain number. Find the number in each case.

(a) 2.8820 (b) $-2 + .111$

Solution

(a) Let y be the number.

We then know $\log_{10}y = 2.8820$

Therefore $y = 10^{2.8820}$

$$\begin{aligned} y &= 10^{.8820} \times 10^2 \\ &= 7.62 \times 10^2 \\ y &= 762 \end{aligned}$$

(b) Let y be the number.

We know $\log_{10}y = -2 + .111$

Therefore $y = 10^{(-2+.111)}$

$$\begin{aligned} &= 10^{-2} \times 10^{.111} \\ &= .01 \times 1.291 \qquad \text{Why?} \\ y &= .01291 \end{aligned}$$

EXERCISE 11·11

1. Write each in logarithmic form.
   (a) $81 = 10^{1.9085}$ (b) $628 = 10^{2.7980}$
   (c) $.0455 = 10^{-2+.6580}$ (d) $.0455 = 10^{-1.342}$
   (e) $.00345 = 10^{-3+.5378}$ (f) $478 = 10^{2+.6794}$
2. In question 1(c) and (d) the exponential form of .0455 is given in two different ways. Explain.

3. Write each in logarithmic form.

(a) 394 (b) 28300 (c) 1.45

(d) 0.00479 (e) 0.0697 (f) 4.17

4. Write in exponential form.

(a) $\log_{10}88 = 1.9445$ (b) $\log_{10}44.8 = 1.6513$

(c) $\log_{10}600 = 2.7782$ (d) $\log_{10}.04 = -2 + .6021$

(e) $\log_{10}.05 = -1.301$ (f) $\log_{10}666 = 2.8235$

5. Each of the following is a logarithm to the base 10 of a number. Find the number in each case.

(a) 3.5944 (b) 1.7042 (c) 0.7694

(d) $-1 + .8751$ (e) $-1 + .4000$ (f) $-2 + .900$

Using the Library

1. *The Mathmen*: Read pages 65-66. Find as many Pythagorean Triples as you can.

   Read Chapter 2, pages 42-73. What does the Pythagorean motto "A theorem and a platform, not a theorem and an obol (penny)" mean?

   Why did the Pythagoreans adapt the pentagram as their distinctive badge?

2. *Mathematics Enrichment*, R. D. Ripley, George E. Tait. The Copp Clark Publishing Company.

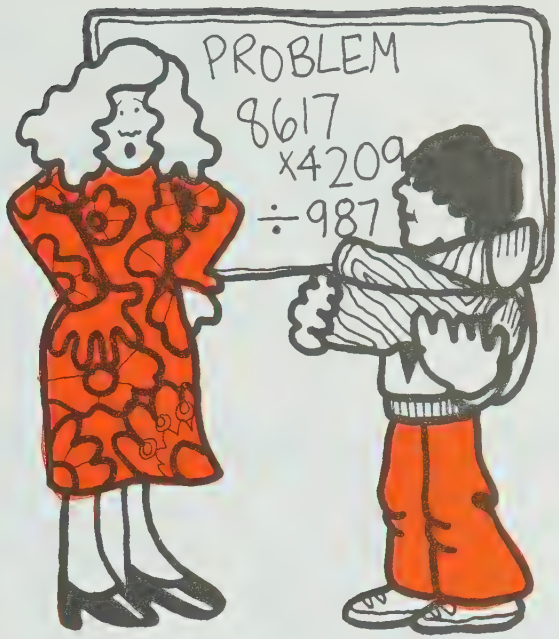
   Read page 10. Use the method Gauss would use to find the sum $1 + 2 + 3 + \quad + 100$.

   Read page 64. Can you find wherein lies the following? Make the rectangles and triangles to see what actually happens.

3. *The Giant Golden Book of Mathematics*. Read pages 28-29. Using compasses and ruler draw a line $\sqrt{3}$ units long. Evaluate without tables $\sqrt{12}$ to two decimal places.

   Read pages 18-20. Write the first 8 triangular numbers; the first 8 square numbers; the first 8 cubic numbers.

   Read page 20. Which payment plan would you take were you the farm boy? Why?



No, Barry! That's not what I meant when I told you to use logs.

REVIEW EXERCISE 11

1. Simplify:

(a) $\sqrt{5} \times \sqrt{6}$ (b) $\sqrt{576}$ (c) $\sqrt{18n^2m^6}$.

2. Write the standard numeral for each:

(a) $\sqrt{13^2}$ (b) $\sqrt{34}$

(c) $\sqrt{2025}$ (d) $\sqrt{16.81}$.

3. Rationalize the denominator. Simplify:

(a) $\dfrac{\sqrt{7}}{\sqrt{2}}$ (b) $\sqrt[3]{\dfrac{40}{3}}$

(c) $\dfrac{\sqrt{m}}{\sqrt{n}}$ (d) $\dfrac{\sqrt{m-n}}{\sqrt{m+n}}$.

4. If $\sqrt{2} = 1.414$ and $\sqrt{3} = 1.732$ find correct to 2 decimal places, the numerical value of each:

(a) $\dfrac{\sqrt{2}}{\sqrt{3}}$ (b) $\dfrac{3}{\sqrt{8}}$ (c) $2\sqrt{3} \times 3\sqrt{6}$.

5. Express each with one positive exponent.

(a) $\frac{5^6}{5^{\frac{2}{3}}}$ (b) $\frac{(\frac{3}{4})^6}{(\frac{3}{4})^4}$ (c) $8^{\frac{2}{3}} \cdot 8^{\frac{3}{4}}$

(d) $a^{\frac{1}{3}} \cdot a^{\frac{5}{6}}$ (e) $(5^2)^3$ (f) $(3^{\frac{1}{2}})^{\frac{5}{6}}$

(g) 16^{-3} (h) $8^{-\frac{3}{4}}$ (i) $(\frac{3}{4})^{-2}$.

6. Evaluate each:

(a) $8^{\frac{1}{3}}$ (b) $81^{\frac{1}{2}}$

(c) $64^{\frac{2}{3}}$ (d) $10000^{\frac{3}{4}}$.

7. A ball is dropped from a height of 10 feet. On each bounce it reaches $\frac{3}{4}$ of its previous height. Make a graph to show the exponential decay for 0 through 8 bounces. Write the exponential function.

8. Use Figure 11-5.

(a) Express as a power of 10.
55 8.2 0.3

(b) Express each in decimal notation.
$10^{.45}$ $10^{1.25}$ $10^{-1+.35}$

(c) Perform the operations.
$7.9 \div 2.5$ 433×6.7

9. Write each in standard notation.
443 51.2 0.076

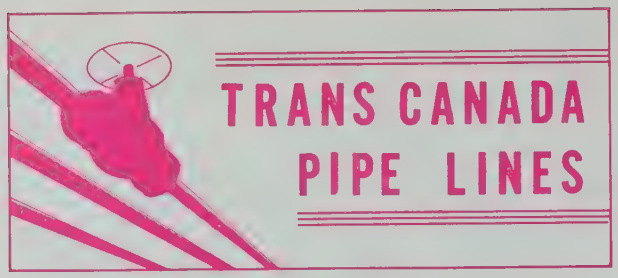
10. Write each in the form 10^x.
$10^{.3} \times 10^{-2}$ $10^{.65} \times 10^{-5}$ $10^{-3} \times 10^{.78}$

11. Write as the product of two powers of 10—one with a positive exponent and the other with a negative exponent.
$10^{-2.3}$ $10^{-1.45}$ $10^{-3.76}$

12. Use the table $y = 10^x$. Write each number in exponential form to base 10.
3.45 0.086 917

13. Write each in decimal notation.
$10^{1.4014}$ $10^{2.7536}$
$10^{-1+.2253}$ $10^{-2+.8808}$

14. Evaluate each. Show your work.
425×235 $856 \div 35$ $\sqrt{88}$



Our company operates a pipe line system for the purpose of transporting crude oil from its point of production in the oil fields to Edmonton and Edson, Alberta. We frequently have to lay new pipe lines which, of course, involves order of a certain number of miles of pipe to connect new fields, etc. Therefore, I have set out two problems below that have to do with this part of our oil transportation business.

PROBLEM No. 1

The length of a new 12"-diameter line is to be 42 miles. The steel mills make the pipe in separate joints, each joint having a length of 57 feet.

It will cost the company $3.83 per foot to purchase the pipe.

Once the pipe is laid it must be filled from end to end with oil and it requires 0.1457 barrels of oil to fill one foot of the line.

The cost of the oil used to fill a line is $2.82 per barrel.

Solve for the following:

1. The line will be ■ feet long.
2. The line will contain ■ joints of pipe.
3. Cost of the pipe for the line will be $■.
4. The new line will require ■ barrels of oil to fill it.
5. The cost of the oil to fill the line will be $■.

PROBLEM No. 2

To make sure that enough pipe would be available to lay the line the company actually ordered 1% more than the 42-mile surveyed distance of the line. Of this 1% overage allowance on the pipe order, 28.2% was actually used.

From this information determine how much pipe the company had left on hand when the job was completed.

Chapter Twelve
LENDING AND BORROWING MONEY

12·1 Simple Interest

Nearly all of us, from time to time, will have occasion to lend or borrow money – in one form or another. We may lend by putting our money in a savings account or in some investment. We receive payment for letting someone use our money. This payment is called *interest*. It may be that we will borrow from a friend (a very poor policy), from a bank or from a finance company. On the other hand, we may borrow indirectly (at least it appears indirect) from a firm where we buy an article *on time* with an agreement to pay either in *instalments* or in a lump sum at the end of a certain period of time. We know we must pay for the use of someone else's money. The amount we pay is called the *interest*.

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EXERCISE 12·1

1. Let us draw graphs to show the relation between the simple interest (I) and the time (t) in years on \$100 at 3% per annum. Use the formula $I = prt$ to complete Table 1.

$$\text{1 year} \begin{cases} I = prt \\ I = 100 \times .03 \times 1 \\ I = \$3 \end{cases}$$

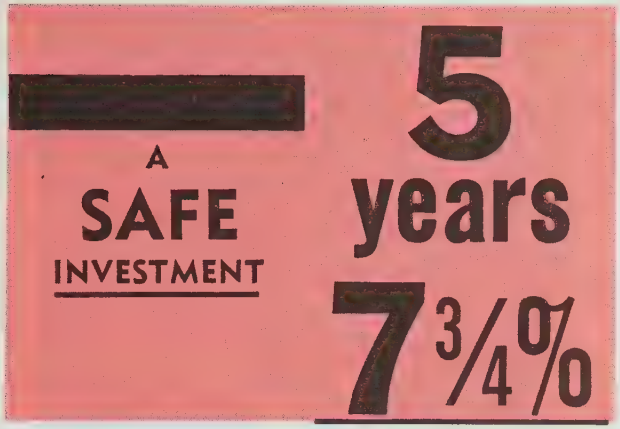
$$\text{2 years} \begin{cases} I = prt \\ I = 100 \times .03 \times 2 \\ I = \$6 \end{cases}$$

$$\text{3 years} \begin{cases} I = prt \\ I = 100 \times .03 \times 3 \\ I = \$\blacksquare \end{cases}$$

Table 1

t	1	2	3	4	5	6	7	8	9	10
I	\$3	\$6								

2. Repeat question 1 for
   (a) 6% (b) 9% (c) 12% (d) 15%.
3. How do amount of interest and time vary?
4. How do interest rate and interest vary?
5. Use the data above to draw a graph showing the relation between interest and rate.
   Hint! Use the data from each chart for the interest for 5 years at the various rates.
6. Which graph illustrates the answer to question 4? Explain.
7. Prepare a table and graph to show the relation between:
   (a) principal and time (rate and interest constant);
   (b) principal and rate (time and interest constant);
   (c) principal and interest (time and rate constant).
8. Describe the relation (type of variation) for each graph in question 7.



12·2 Interest: Time in Days

In working with elementary interest problems the time is usually given in years. There are many cases in life-situations in which the time is calculated in days. However, the formula requires time t to be stated in years. Thus, if 65 days is the time of the loan then $\frac{65}{365}$ is the replacement for t in the formula since 65 days is $\frac{65}{365}$ years.

Example 1

A loan of \$650 is made on February 10 and paid on April 24. Find the interest if the rate is $11\frac{3}{4}\%$ per annum.

Solution

First, the time in days must be found. By agreement in business circles the first day of the period is not counted.

Days in February	18
Days in March	31
Days in April	24
Total	73 days

Now find the interest

$$I = prt$$

$$I = 650 \times .1175 \times \frac{73}{365}$$

$$= \$15.28 \quad \text{(to the nearest cent)}$$

EXERCISE 12·2

1. Find the interest on \$500 invested for 35 days at a rate of 8% per annum.

2. Find the interest on \$757 invested for 195 days at a rate of 12% per annum.

3. John Marcos obtains a loan on May 23. He pays the loan up on September 17 of the same year. What is the time of the loan in days? in years?

4. A fisherman obtains a short term loan from a bank on March 3. The amount of the loan was \$645. The rate was 15% per annum. What was the interest on the loan if he repaid it on August 1? How much did he pay back altogether?

5. Find the amount needed to cover a loan of \$1,450 obtained on November 19, 1971, at $11\frac{1}{2}\%$ per annum, and repaid on March 15, 1972. (1972 is a leap year.)

12·3 Financial Calendars

Banks and financial institutions must make many calculations each day similar to those in the questions of the preceding exercise. To assist the employees to calculate the number of days between any two calendar dates a special calendar is prepared. The calendar in Table 2 is used. Study the calendar carefully.

Table 2

Day of Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	351
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	323	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

The use of this table is illustrated in the two examples.

Example 1

Find the number of days on which the interest would be calculated if the loan were made on February 26 and repaid on October 6 of the same year.

"Got it now? During any sharp advance you circulate among the customers booming, 'Ho ho ho!'"

Solution

From the table we see that February 26 is the 57th day and October 6 is the 279th day. Therefore, 279 – 57 = 222 is the number of days on which interest is paid. (Provided it is not a leap year in which case one extra day would be added.)

Example 2

Find the number of days between September 16, 1971 and March 26, 1972.

Solution

September 16 to December 31 is:

$$365 - 259 = 106 \text{ days}$$

December 31 to March 26 is:

$$85 - 0 = 85$$

But 1972 is a leap year, therefore February has 1 more day:

$$85 + 1 = 86 \text{ days}$$

$$\text{Total days} = 192$$

Notice we counted December 31 in the first period and that we designate December 31 as 0 in the second period. Since January 1 is day 1, then December 31 can be considered day 0.

EXERCISE 12·3

Find the number of interest days for the periods (years are nonleap years unless indicated otherwise).

1. April 1 to July 15 of the same year.
2. February 15 to September 20 of the same year.
3. July 13 to December 15 of the same year.
4. December 24 to January 6 of the next year.
5. December 10 to January 12 of the next year.
6. October 17 to February 14 of the next year.
7. November 1 to May 6 of the next year (leap year).
8. May 2 to May 1 of the next year.
   Can you determine a short way to calculate this?
9. Find the interest on a loan of \$9,550 obtained on March 23 and repaid on November 10. The interest rate is $9\frac{3}{4}\%$ per annum.
10. Calculate the amount necessary to repay \$756 loan with an interest rate of $11\frac{1}{4}\%$ per annum. Date of loan is September 26 and repayment date is March 13 the next year.
11. Calculate the interest on a loan of \$1,360 obtained on November 23 and repaid February 15 the next year. The interest rate is $8\frac{1}{2}\%$ per annum.

12·4 Step Functions: Simple Interest

Joseph invests \$150 in a company for an indefinite amount of time with the agreement that he will only receive interest for any one year if he leaves the money in the company for that whole year. It is agreed he will receive $6\frac{1}{2}\%$ per annum.

Let us find the amount of interest he will receive for various lengths of time and show it on a graph.

Under the agreement he will not receive any interest unless he leaves the money in at least one year.

If he leaves the money in more than one year but less than 2 years he receives interest for one year only. The graph of the interest from one year to one day less than 2 years is shown in Figure 12-1.

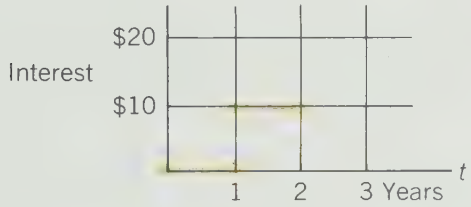


Figure 12-1

He can leave the money in for 3 years less one day and still receive only two lots of interest of \$9.75 each lot. (See Figure 12-2.)

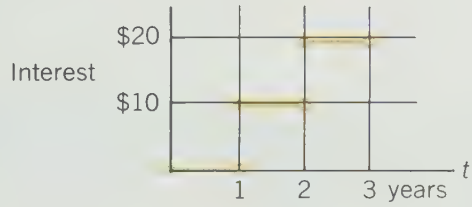


Figure 12-2

If we continue the graph for 5 years we get the graph in Figure 12-3.

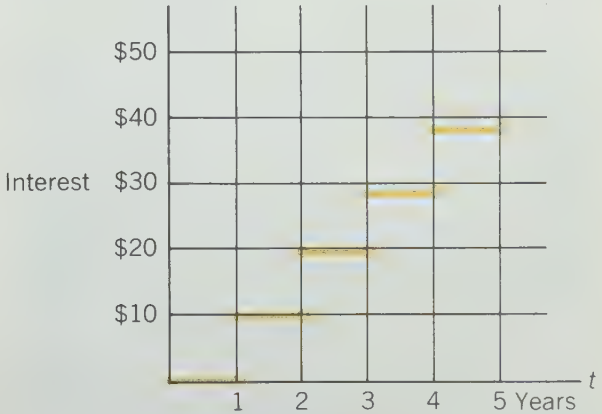


Figure 12-3

This is called a step function. What is the amount of interest he would receive if he left the money invested for 3 years 10 months and 2 days? 1000 days? 730 days?

EXERCISE 12·4

1. There are many step functions. Show by means of a graph
   (a) the relation between the cost and number of 5¢ stamps;
   (b) the relation between the number of birthdays and the number of years since birth.

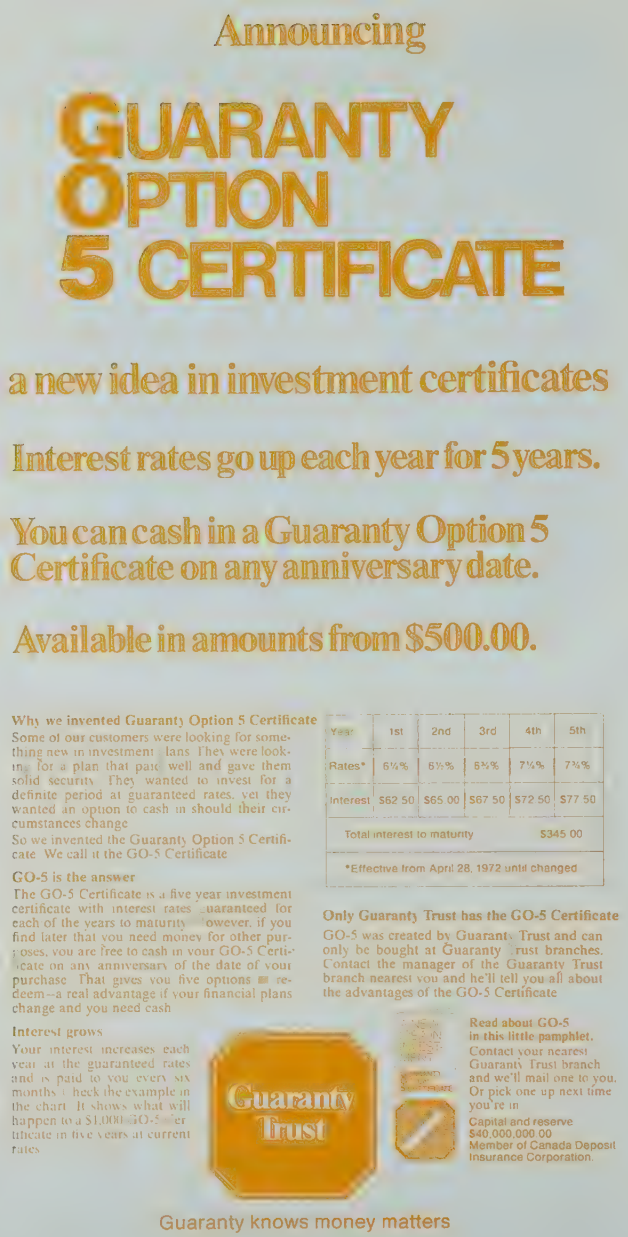
2. Suppose taxi rates are \$.50 for the first $\frac{1}{5}$ of a mile plus 10¢ per $\frac{1}{5}$ mile thereafter. Draw a table of values to show the relation between cost and miles.

miles	0	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	1
cost	.50	.50								

Draw a graph to show the relation.

3. Show by means of a graph the simple interest a person would receive on \$500 at 9% per annum if it were agreed that he would receive interest for any given year only if he left the money invested for that full year.

4. Show by means of a graph the simple interest a person pays on \$300 at 11% per annum if it is agreed he will pay interest for any year if he has the money for any part of that year.


Announcing
GUARANTY OPTION 5 CERTIFICATE
a new idea in investment certificates
Interest rates go up each year for 5 years.
You can cash in a Guaranty Option 5 Certificate on any anniversary date.
Available in amounts from $500.00.
Why we invented Guaranty Option 5 Certificate
Some of our customers were looking for something new in investment plans. They were looking for a plan that paid well and gave them solid security. They wanted to invest for a definite period at guaranteed rates, yet they wanted an option to cash in should their circumstances change.
So we invented the Guaranty Option 5 Certificate. We call it the GO-5 Certificate.
GO-5 is the answer
The GO-5 Certificate is a five year investment certificate with interest rates guaranteed for each of the years to maturity. However, if you find later that you need money for other purposes, you are free to cash in your GO-5 Certificate on any anniversary of the date of your purchase. That gives you five options to redeem—a real advantage if your financial plans change and you need cash.
Interest grows
Your interest increases each year at the guaranteed rates and is paid to you every six months. Check the example in the chart. It shows what will happen to a $1,000 GO-5 Certificate in five years at current rates.
Year 1st 2nd 3rd 4th 5th
Rates* 6¼% 6½% 6¾% 7¼% 7¾%
Interest $62.50 $65.00 $67.50 $72.50 $77.50
Total interest to maturity $345.00
*Effective from April 28, 1972 until changed
Only Guaranty Trust has the GO-5 Certificate
GO-5 was created by Guaranty Trust and can only be bought at Guaranty Trust branches. Contact the manager of the Guaranty Trust branch nearest you and he'll tell you all about the advantages of the GO-5 Certificate.
Guaranty Trust
Read about GO-5 in this little pamphlet.
Contact your nearest Guaranty Trust branch and we'll mail one to you. Or pick one up next time you're in.
Capital and reserve $40,000,000.00
Member of Canada Deposit Insurance Corporation.
Guaranty knows money matters

What is the average interest rate per annum over 5 years if \$1,000 is invested in this plan?

5. Show by means of a graph the amount of the investment in the situation described in question 3.

6. Show by means of a graph the amount payable at any time in question 4.

12·5 Using a Formula to Find Amount

Jacques agrees to lend his brother Emile $250 for one year at an interest rate of 6% per annum. What is the amount that Jacques will receive when Emile pays him one year later?

You know that Emile must pay back the $250 as well as the interest. Therefore we could write:

(1) Amount = principal + interest
We know the principal is $250
Interest can be calculated. It is $15.
Therefore

$$\text{Amount} = \$250 + \$15$$
$$= \$265$$

Alternatively, let us use symbols for each part of statement (1).

$$\text{Amount} = A$$
$$\text{Principal} = p$$
$$\text{Interest} = I$$

We now have

(2) $$A = p + I$$

But we have an expression for I. It is prt.
Let us use prt in place of I

(3) $$A = p + prt$$

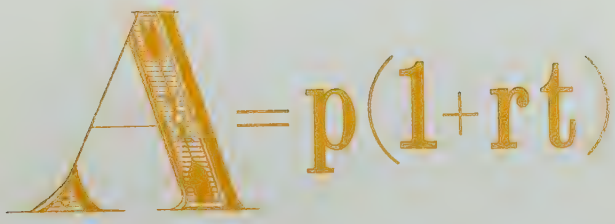
Now using the distributive property we get

$$\boxed{A = p(1 + rt)}$$

Making appropriate replacements we get:

$$A = 250\,(1 + .06 \times 1)$$
$$= 250\,(1.06)$$
$$A = \$265$$

Jacques will receive $265 one year later.



EXERCISE 12·5

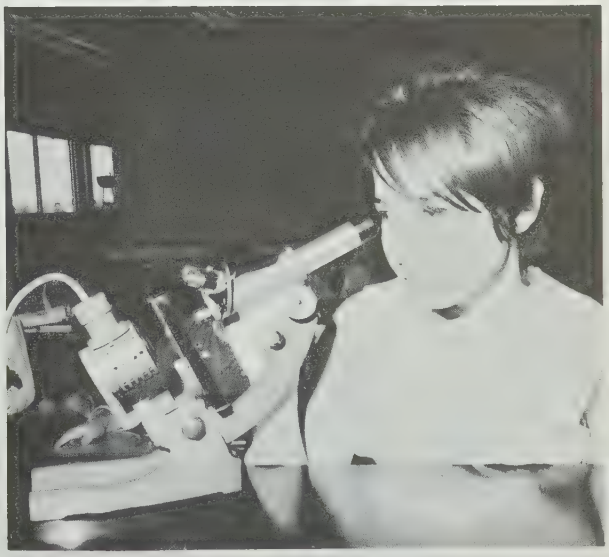
1. Work the long way then verify by using the formula $A = p(1 + rt)$.

   To buy a car Rod borrows $175 from his father and agrees to pay him back two years later. He agrees to pay simple interest at 5% per annum. How much will he pay his father two years later?

2. Find the missing quantity in each problem.

	Amount	*Principal*	*Rate*	*Time (in years)*
(a)	$210	$200	?	1
(b)	$825	?	2%	5
(c)	?	$375	$3\frac{1}{2}$%	10
(d)	$890	$800	3%	?
(e)	$1814.85	?	6.25%	10
(f)	?	$3495	$7\frac{1}{8}$%	6.25

3. Joan plans to attend Technical School two years from now. Her father asked her to calculate how much must be deposited now in a special savings account at $8\frac{7}{8}$% so that $3000 will be available when Joan enrolls in Tech.

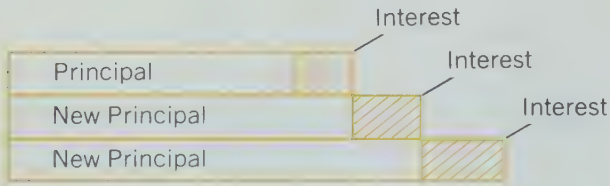


4. John wishes to invest at $7\frac{3}{4}$% an amount today such that in 5 years he will have $1000. How much must he invest?

12·6 Compound Interest

In the previous chapter we discussed growth patterns of bacteria, area, volume and prize money. We reserved the balance of this chapter for the growth of an investment as interest accumulates.

The difference between compound interest and simple interest is that in simple interest the interest is always calculated on the basis of one principal that remains constant. In compound interest the principal changes at the end of each interest period. The new principal is always the old principal plus the interest earned: the principal always gets larger after each interest period.



ROYAL TRUST
GUARANTEED INVESTMENT RECEIPTS

Are fully guaranteed as to principal and interest by the Company.

- The rate at which the G.I.R. is issued is guaranteed for the term of the deposit.
- G.I.R.'s are fully registered and interest is paid at maturity on receipts issued for periods up to one year. On all other receipts, interest is paid semi-annually in June and December, and at maturity.
- Deposits for less than one year may be withdrawn at any time in whole or in part subject to an adjustment in the interest rate. Particulars may be obtained by contacting the Company.
- There is no fee or handling charge.

*

INTEREST RATES EFFECTIVE 11th MARCH, 1970
5 YEARS

9%

TERM	RATE	TERM	RATE
Demand	on request		
30 - 59 days	— 7.75 %	180 - 364 days	— 8.25 %
60 - 89 days	— 7.875%	1 year and over	— 8.50 %
90 - 179 days	— 8.125%	3 years and over	— 8.50 %

Full 5 year term — 9%

*Minimum Deposit: Under 1 year $5,000 - 1 year and over $1,000 - 5 years only $500.00

ROYAL TRUST
CANADA'S LARGEST TRUST COMPANY
ASSETS UNDER ADMINISTRATION
$10 BILLION

Advertisement which appeared during March, 1970.

Example 1

A man invests $1000 at an interest rate of 10% per annum. The interest due each year is added to the principal and the interest in the next year is calculated on this new principal which is the sum of the previous principal and the interest. Find the amount the man has to his credit at the end of three years.

Solution

Principal invested (at beginning of first year)	$1000
Interest at end of first year: 10% of $1000	100
Principal at beginning of second year	$1100
Interest at end of second year: 10% of $1100	110
Principal at beginning of third year	$1210
Interest at end of third year: 10% of $1210	121
Amount of the investment at end of third year	$1331

EXERCISE 12·6

Follow the method set in Example 1 to calculate the amount of each loan.

	Principal	*Interest rate*	*Time*
1.	$2000	9% compounded annually	4 years
2.	$5000	8% compounded annually	3 years
3.	$3500	$7\frac{1}{2}$% compounded annually	5 years
4.	$4250	$6\frac{1}{4}$% compounded annually	3 years

5. (a) Compute the amount at the end of each year for 10 years of $1000 invested at 10% compounded annually. Plot this information on a Cartesian graph in which the horizontal axis is the time in years and the y-axis is the amount in dollars. Draw a continuous line to contain the points on the graph.

   (b) With reference to the same set of axes (in (a)) draw a line (continuous) graph showing the amount at simple interest of 10% annually on the $1000 for 10 years

   (c) How long does it take $1000 to become $2000 at the interest rate of 10% per annum compounded annually?

   (d) How long does it take for $1000 to double itself at 10% per annum simple interest?

12·7 Amount Formula: $A = (1 + i)^n$

Let us now generalize the problem in Example 1 of the preceding section.

Principal invested: $1000

Interest at end of first year: $1000 \times .10$

Principal at beginning of second year:

$$1000 + 1000 \times .10 = 1000\,(1 + .10)$$
$$= 1000\,(1.10)$$

Interest at end of second year: $1000\,(1.10)(.10)$ *Why?*

Principal at beginning of third year:

$$1000\,(1.10) + 1000\,(1.10)(.10)$$
$$= 1000\,(1.10)(1 + .10)$$
$$= 1000\,(1.10)(1.10)$$
$$= 1000\,(1.10)^2$$

Interest at end of third year: $1000\,(1.10)^2(.10)$

Principal at beginning of fourth year:

$$1000\,(1.10)^2 + 1000\,(1.10)^2(.10)$$
$$= 1000\,(1.10)^2(1 + .10)$$
$$= 1000\,(1.10)^3$$

From this we see that the amount of the investment after 3 years is

$$1000(1.10)^3$$

Generalizing, the amount after n years would be

$$1000(1.10)^n$$

Generalizing still further, if the principal was $\$p$, the compound interest rate was i and the time n years, then the amount ($\$A$) would be

$$A = p(1 + i)^n$$

Example 1

Find the amount of $5000 invested at 12% compounded annually for 3 years.

Solution

Using the formula:

$$A = p(1 + i)^n$$

we get $A = 5000(1.12)^3$.

Calculating $(1.12)^3$ we get

$$(1.12)(1.12)(1.12) = 1.40493 \text{ (to 5 decimal places)}$$
$$A = 5000(1.40493)$$
$$= \$7{,}024.65$$

In Section 12·6 we showed that $1000 invested at 10% compounded annually for 3 years is $1331. To show the difference between simple and compound interest we present the same problem calculated at 10% simple interest.

$$A = p + p \times i \times n$$
$$= 1000 + 1000 \times .10 \times 3$$
$$= 1300$$

There is a difference of $31 – $31 more in 3 years!

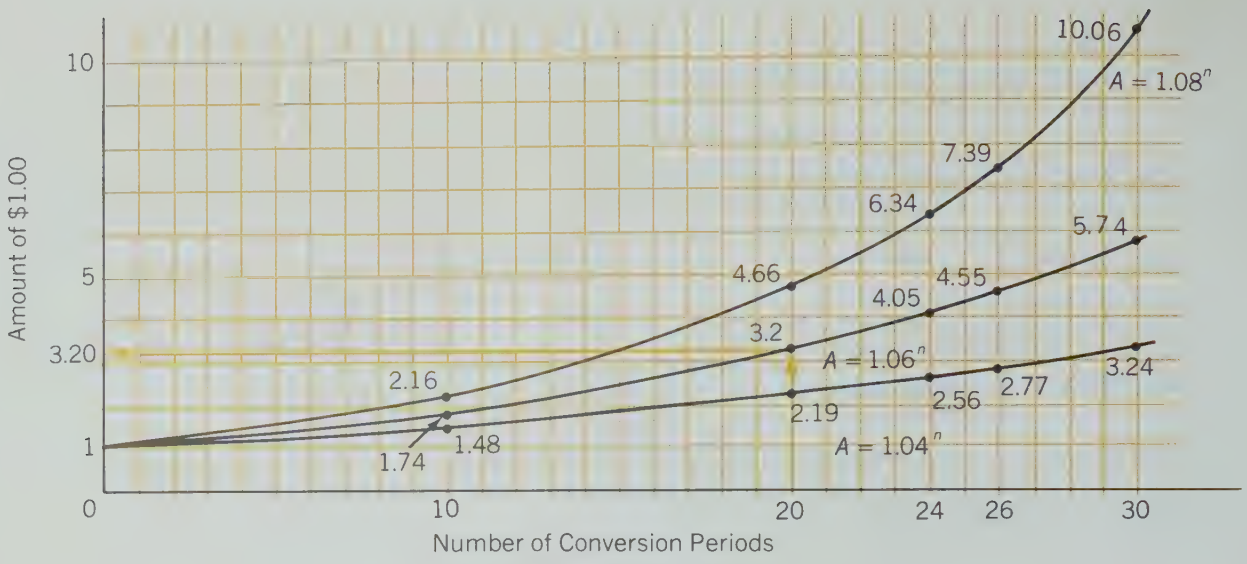


Figure 12-4

Figure 12-4 shows the amount of $1.00 invested at 4%, 6% and 8% for values of n from 0 to 30 years.

Example 2

From the graph determine the amount of $200 after 20 years compounded annually at 6%.

Solution

From the graph we shall find the amount of $1.00 after 20 years at 6%. To do this follow the arrow from $n = 20$ to the graph of $A = 1(1.06)^n$ then horizontally to the A where we find the value of $3.20 (approximately).

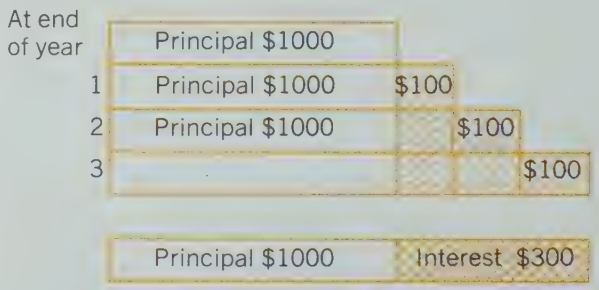
Hence, $1.00 amounts to $3.20 (approximately)

Therefore, $200 amounts to $640 (approximately)

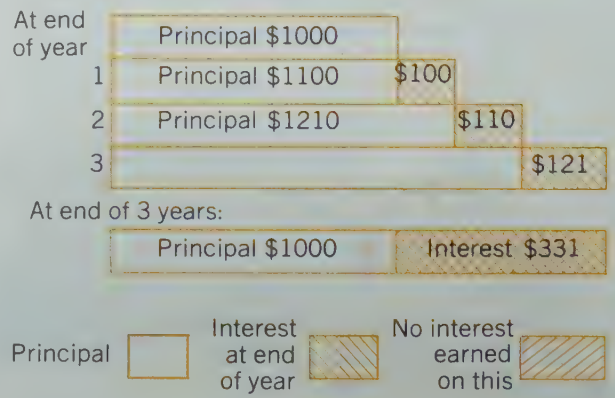
Computers are usually used to solve interest problems.

Let us compare simple interest and compound interest pictorially.

Simple Interest



Compound Interest



EXERCISE 12·7

1. Use the formula to find the amount in each case.

	Principal	Time (in years)	Interest rate (compounded yearly)
(a)	$400	3	15%
(b)	$650	2	9%
(c)	$730	5	10%

2. A man debates whether to invest $1000 for 3 years at
   (a) $8\frac{1}{4}\%$ simple interest or
   (b) 8% compounded annually

   Which offer is the better, and how much can the man gain by making the proper choice?

3. J. Brown, a retired farmer, debates whether to invest his $50,000 at 7% compounded annually for 4 years or at $7\frac{3}{8}\%$ simple interest. How much can he gain by making the proper choice?

4. Use Figure 12-4 to find the amount in each (compounded annually).

	Principal	Time (in years)	Rate	Amount
(a)	$300	10	8%	
(b)	$350	25	4%	
(c)	$750	15	4%	
(d)	$500	20	8%	
(e)	$100	30	6%	

5. A man invests $100 at 6% compounded annually for his daughter when she is born. How much will she have on her 21st birthday?

12·8 Compounding Interest More Often Than Yearly

The rate of interest is always quoted on the annual basis. Yet it is very common to compound the interest more often than yearly. Therefore, we must be able to equate interest on a yearly basis to interest on a semiannual, quarterly or monthly basis.

The interest on a semiannual basis is half of that on an annual basis. If the annual basis is 6% then for a semiannual basis it is half of 6% or 3%. On a quarterly basis it is $\frac{6}{4}\%$ and on a monthly basis it is $\frac{6}{12}\%$ or half of 1%. In compounding interest, if it is compounded quarterly, there are four *conversion periods* of $1\frac{1}{2}\%$ each instead of one conversion period at 6%.

Example 1

Find the interest rate for the conversion period if the annual rate and compounding periods are

(a) 8% quarterly (b) 12% monthly

Solution

(a) 8% annually compounded quarterly is 2% per conversion period (8 ÷ 4).

(b) 12% annually compounded monthly is 1% per conversion period (12 ÷ 12).

Example 2

Find the number of conversion periods in

(a) 3 years, compounded semiannually

(b) $5\frac{1}{2}$ years, compounded monthly.

Solution

(a) Semiannually means 2 periods per year, therefore, in 3 years there are 3×2 or 6 periods.

(b) Monthly indicates 12 periods per year so in $5\frac{1}{2}$ years there are $5\frac{1}{2} \times 12$ or 66 periods.

Example 3

$2000 is invested at 8% compounded semiannually. Calculate the amount of this investment after 2 years.

Solution

There are 4 interest periods. 8% per annum compounded semiannually is equivalent to 4%.

Principal invested	$2000
Interest at end of 6 months ($.04 \times 2000$)	80
Principal at beginning of second 6 month period	2080
Interest at end of second 6 month period ($.04 \times 2080$)	83.20
Principal at beginning of third 6 month period	2163.20
Interest at end of third 6 month period ($.04 \times 2163.20$)	86.53
Principal at beginning of fourth 6 month period	2249.73
Interest at end of fourth 6 month period ($.04 \times 2249.73$)	89.99
Amount at end of 2 years	$2339.72

There are certain expressions we will use in this section. The following terms are important.

Conversion Period is the time elapsing between two successive calculations of interest and the addition of this interest to the principal.

Term is the length of time of the loan.

Compound Interest is the total interest earned at the end of the term of the loan.

Compound Amount is the total amount due at the end of the term of the loan.

The formula for compound amount used earlier will be used in this section also. In general, in $A = p(1+i)^n$:

A represents the compound amount in dollars.

p represents the principal in dollars.

i represents the rate of interest per period expressed as a decimal numeral.

n represents the number of interest periods.

Example 4

Find an expression for the compound amount of $1250 at 10% per annum compounded quarterly for a time of 5 years.

Solution

Using the formula $A = p(1 + i)^n$ we first calculate i and n. Since the interest is compounded quarterly we calculate 10% per annum compounded quarterly: $\frac{10}{4}\%$ or 2.5% per period. Five years is 20 conversion periods.

$$A = p(1 + i)^n$$
$$= 1250\,(1 + .025)^{20}$$

$$A = 1250\,(1.025)^{20}$$

We realize it would be a great deal of work even with a calculator to evaluate $(1.025)^{20}$. To assist in the calculation of compound amount, tables have been established. In the appendix of this book a table may be found that gives the amount of $1 at various rates of interest for various periods. An excerpt from this table is shown on page 227.

COMPOUND AMOUNT OF 1

YRS.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%
1	1.01000	1.01250	1.01500	1.01750	1.02000	1.02500	1.03000	1.03500
2	1.02010	1.02516	1.03023	1.03531	1.04040	1.05063	1.06090	1.07123
3	1.03030	1.03797	1.04568	1.05342	1.06121	1.07689	1.09273	1.10872
4	1.04060	1.05094	1.06136	1.07185	1.08243	1.10381	1.12551	1.14752
5	1.05101	1.06408	1.07728	1.09062	1.10408	1.13141	1.15927	1.18769
6	1.06152	1.07738	1.09344	1.10970	1.12616	1.15969	1.19405	1.22926
7	1.07214	1.09085	1.10985	1.12912	1.14869	1.18869	1.22987	1.27228
8	1.08286	1.10449	1.12649	1.14888	1.17166	1.21840	1.26677	1.31681
9	1.09369	1.11829	1.14339	1.16899	1.19509	1.24886	1.30477	1.36290
10	1.10462	1.13227	1.16054	1.18944	1.21899	1.28008	1.34392	1.41060
11	1.11567	1.14642	1.17795	1.21026	1.24337	1.31209	1.38423	1.45997
12	1.12683	1.16075	1.19562	1.23144	1.26824	1.34489	1.42576	1.51107
13	1.13809	1.17526	1.21355	1.25299	1.29361	1.37851	1.46853	1.56396
14	1.14947	1.18995	1.23176	1.27492	1.31948	1.41297	1.51259	1.61869
15	1.16097	1.20483	1.25023	1.29723	1.34587	1.44830	1.55797	1.67535
16	1.17258	1.21989	1.26899	1.31993	1.37279	1.48451	1.60471	1.73399
17	1.18430	1.23514	1.28802	1.34303	1.40024	1.52162	1.65285	1.79468
18	1.19615	1.25058	1.30734	1.36653	1.42825	1.55966	1.70243	1.85749
19	1.20811	1.26621	1.32695	1.39044	1.45681	1.59865	1.75351	1.92250
20	1.22019	1.28204	1.34686	1.41478	1.48595	1.63862	1.80611	1.98979

Example 5

Find the compound interest and compound amount of \$500 at 7% per annum compounded semiannually for a term of 10 years.

Solution

$$A = p(1 + i)^n$$

$n = 10 \times 2 = 20$ periods

$i = 7 \times \frac{1}{2} = 3\frac{1}{2}\%$

$$A = 500\,(1.035)^{20}$$

To evaluate $(1.035)^{20}$ find 20 in the "Number of Periods" column (column marked YRS. in the table), and move horizontally across to the vertical column marked $3\frac{1}{2}$. You find 1.98979. This means that \$1 under the conditions described would amount to \$1.98979.

Hence, \$500 would amount to 500 times as much

$A = 500\,(1.98979)$ $[(1.035)^{20} = 1.98979]$
$= 994.895$
$= \$994.90$

The amount is \$994.90 and the interest is

$$\$994.90 - 500 = \$494.90$$

Notice from Example 5 that at 7% per annum compounded semiannually, a given amount of money (\$1) nearly doubles itself in 10 years (\$1.99). Under simple interest at 7% the amount after 10 years of \$1 is \$1.70. It takes nearly 15 years to double a given amount of money by simple interest at 7%.

EXERCISE 12·8

Find the rate of interest per conversion period in each instance.

1. 6% per annum compounded semiannually
2. 12% per annum compounded quarterly
3. 18% per annum compounded monthly
4. 8% per annum compounded semiannually
5. 9% per annum compounded monthly

Find the number of conversion periods in each instance.

6. term of 3 years compounded quarterly
7. term of 8 years compounded semiannually
8. term of $14\frac{1}{2}$ years compounded monthly

Use the Compound Interest Table in the appendix to evaluate each.

9. $(1.02)^8$
10. $(1.05)^{12}$
11. $(1.035)^{15}$
12. $(1.015)^{30}$
13. $(1.025)^{28}$
14. $(1.045)^{52}$

With the aid of the Compound Interest Table find the compound amount in each of the following.

15. \$550 in 5 years at 8% per annum compounded semiannually.
16. \$6300 in 8 years at 10% per annum compounded quarterly.
17. \$280 in 3 years at 18% per annum compounded monthly.

With the aid of the Compound Interest Table find the compound interest in each.

18. $3000 in 3 years at 7% per annum compounded semiannually.

19. $4500 in 2 years at 12% per annum compounded monthly.

20. $720 in 6 years at 12% per annum compounded quarterly.

21. $200 in 4 years at 10% per annum compounded every three months.

22. Mr and Mrs. Fairview invest $100 on the birth of their first child. They plan to present the child with the proceeds of the investment on the child's 21st birthday. If the money is invested at 12% per annum compounded yearly, how much will the child receive as a 21st birthday present?

Project

Project 1

Arrange to visit a bank.

Banks use a *step function* method of calculating interest. What does this mean?

How often does a bank calculate interest on the different types of accounts?

How often does a bank *enter* the interest on the different types of accounts? (What does *enter* mean?)

What is the current rate on the different types of accounts?

12·9 Beyond the Table and Partial Periods

The Table of Compound Interest in the appendix is good up to 40 interest periods. We shall now learn how to calculate the compound amount of the investment beyond 40 periods.

Example 1

Find the compound amount of $300 at 12% per annum compounded monthly for 5 years.

Solution

In 5 years there are 60 conversion periods. Since the table only shows 40 we shall find the compound amount at the end of 40 conversion periods, then using this amount as new principal we shall calculate the new compound amount for another 20 conversion periods.

First Period of 40 Conversion Periods

$A = p(1 + i)^n$ $\quad p = \$300;\ i = .01;\ n = 40$

$= 300\ (1.01)^{40}$

$= 300\ (1.48886)$

$= \$446.658$

Second Period of 20 Conversion Periods

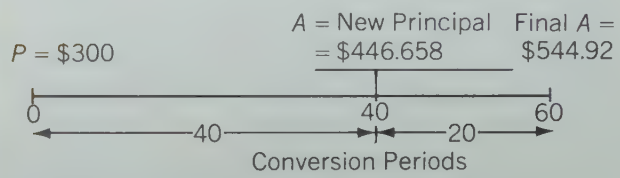
$A = p(1 + i)^n$ $\quad p = \$446.658;\ i = .01;\ n = 20$

$A = 446.658\ (1.01)^{20}$

$= 446.658\ (1.22019)$

$= \$544.92$

The compound amount is $544.92



Sometimes interest is paid on parts of a conversion period. In this case the interest is compounded in the normal way for all the full conversion periods, then the interest for the balance of the time is calculated as simple interest on the compound amount as the principal.

Example 2

Find the compound amount of $500 at 6% per annum compounded semiannually for 3 years 3 months.

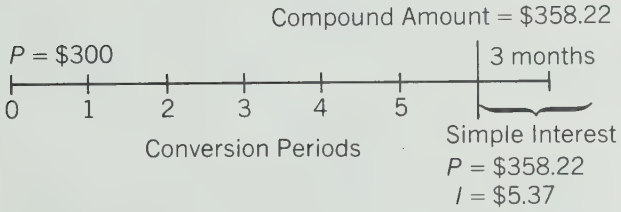
Solution

Step 1. Find the compound amount for 3 years

$$A = p(1 + i)^n$$
$$= \$300\,(1.03)^6$$
$$= 300\,(1.19405)$$
$$= \$358.22$$

Step 2. Calculate the simple interest on $358.22 at 6% per annum for 3 months.

$$I = prt$$
$$= 358.22 \times .06 \times \tfrac{3}{12}$$
$$= \$5.37$$



Step 3. Add this interest to the compound amount.

Final amount = compound amount + interest
$$= 358.22 + 5.37$$
$$= \$363.59$$

Alternate method:

Step 2. Calculate the amount at simple interest on $358.22 at 6% per annum for 3 months.

$$A = p(1 + rt)$$
$$= 358.22\,(1 + .06 \times \tfrac{3}{12})$$
$$= 358.22\,(1.015)$$
$$= \$363.59$$

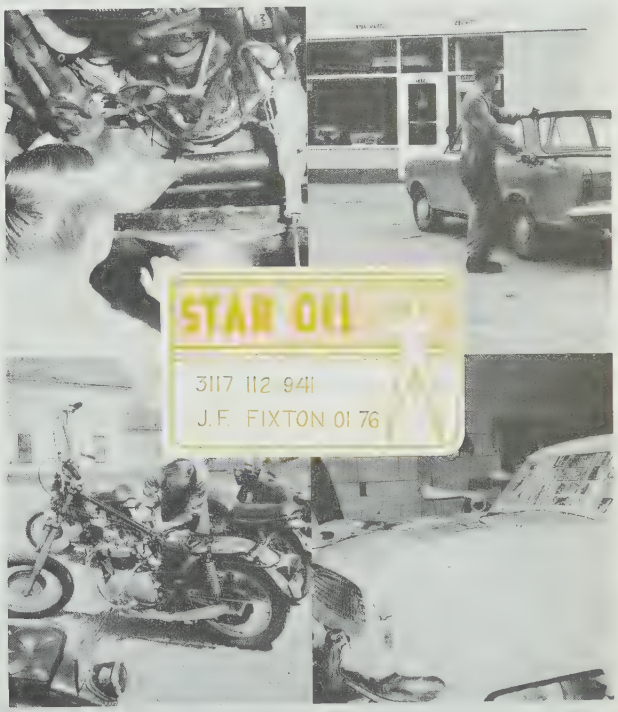
The compound amount is $363.59.

EXERCISE 12·9

1. A young forward-looking man invests his first pay check of $300 at 8% per annum compounded semiannually. Forty-five years later he withdraws his accumulated wealth and goes on a holiday. What is the compound amount of his investment?

2. Mr. and Mrs. Fairview invest $100 on the birth of their second child. The investment is made at 12% per annum compounded monthly. How much will the investment amount to on the child's 21st birthday? Compare this to question 22, Exercise 12·8.

3. Many of the oil companies that issue consumer credit cards charge "18% per annum compounded monthly on unpaid accounts". This is sometimes advertised as "$1\frac{1}{2}$% per month of the unpaid account". Are the two statements equivalent? Which rate sounds the lower to the uneducated public?



4. Charles, who has charged $84.50 worth of gas and car repairs, suddenly loses his job and is unable to pay his credit card account. After 3 months he obtains a new job and is able to pay his bill. How much would he have to pay (using the interest rate stated in question 3)?

5. A man borrows $500 at 6% per annum compounded semiannually. After 5 years 2 months he repays the loan. How much must he repay?

Calculate the compound amount in each case.

6. $450 in 3 years 1 month at 8% per annum compounded quarterly.

7. $150 in 30 years at 9% per annum compounded semiannually.

8. $740 in 35 years at 8% per annum compounded quarterly.

9. List valid reasons one might borrow money on a long term basis.

10. W. Parker Mauldin predicts the rapidly increasing quantity of life threatens the quality of human life. The term population explosion suggests an enormous rate of growth. Yet population is increasing at only 2 percent a year. But the 2 percent is compounded. The amount formula for compound interest [$A = p(1 + i)^n$] can be used to calculate the population in the year 2000. The population in 1970 was 3.5 billion. Consider the period of conversion as one year.

    (a) Calculate the population in the year 2000.

    (b) In what year will the population be doubled (about 6.8 billion)?

    (c) If the population growth were 3 percent compounded yearly, how many years would it take to double the population?

12·10 Present Value

So far we have been calculating the amount that a sum of money will become after a certain period of time. We shall now ask and deal with questions of this type:

What sum of money invested at 4% compounded annually will provide $1000 in 10 years?

We can use the Amount formula:

$$A = p(1 + i)^n$$

In this case

$A = \$1000$ $\quad p = ?$ $\quad i = .04$ $\quad n = 10$

$$\therefore \quad 1000 = p(1.04)^{10}$$

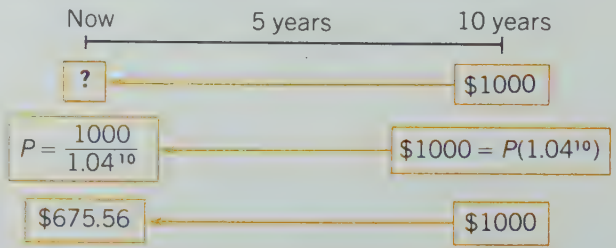
$$p = \frac{1000}{(1.04)^{10}}$$

From the Amount Tables we find $(1.04)^{10} = 1.48024$.

$$p = \frac{1000}{1.48024}$$

$$= \$675.56$$

We must invest $675.56 today in order to have $1000 in 10 years if the interest rate is 4% compounded annually. We can show this on a time line diagram.



The amount that must be invested now is called the *Present Value.*

The formula $A = p(1 + i)^n$

solved for p gives $p = \frac{A}{(1 + i)^n}$.

Since we will call p the present value let us use the symbol *P.V.* instead of p.

Therefore $\boxed{P.V. = \frac{A}{(1 + i)^n}}$

This is called the *Present Value Formula.*

Example 1

A man wishes to invest today enough to pay for his son's college education. It is calculated the son will need $5000 20 years from now. The father can invest at 6% compounded annually. Find the amount (called present value) that the father must invest today.

Solution

Using the present value formula, we have

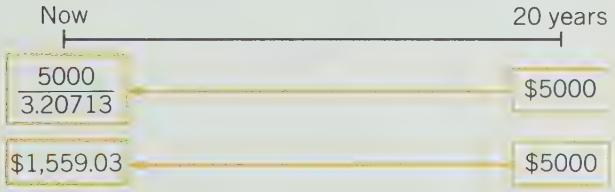
$$P.V. = \frac{A}{(1+i)^n}$$

$$P.V. = \frac{5000}{(1.06)^{20}}$$

$$P.V. = \frac{5000}{3.20713} \qquad \text{(From Amount Tables)}$$

$$P.V. = \$1559.05.$$

The father must invest $1559.05 in order to have $5000 20 years from now.



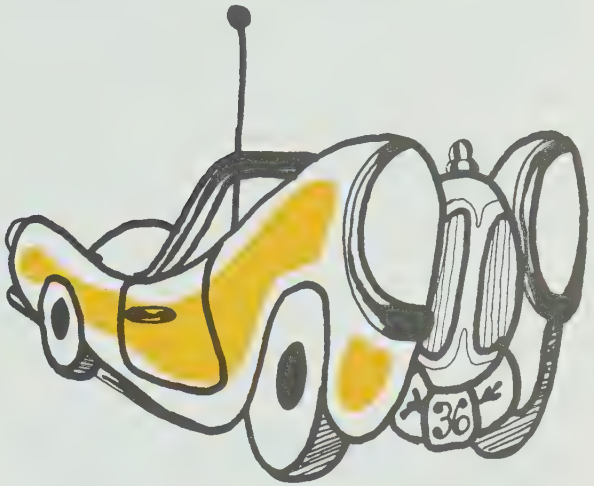
EXERCISE 12·10

1. What amount invested today at 4% compounded annually will have a value of $10,000 in 20 years?

2. Find the present value in each case.

	Amount	*Time*	*Rate per year*	*Conversion scheme*
(a)	$1000	5 years	5%	annually
(b)	$2000	10 years	6%	semiannually
(c)	$5000	8 years	8%	quarterly
(d)	$10,000	6 years	6%	quarterly

3. A sports car enthusiast, on the birth of a son, wishes to invest enough so that the son on his 17th birthday can buy a sports car priced at $5000. The rate of interest is 6% compounded annually. How much must the sports car enthusiast invest?



4. A man plans to retire in 10 years. At the time he retires he wishes to have $6000 to buy a small fishing boat. He is able to invest his money at 6% compounded semiannually. Find the present value of the $6000.

5. Wilfrid Wallace has two offers for his farm. One is for $150,000 now and the other is for $50,000 now and $110,000 in two years. Money is worth 6% compounded annually. Which is the better offer?

12·11 Dated Value

Consider the statement: "Colleen owes $200". This statement merely indicates that Colleen owes some money—actually the amount is vague. To illustrate why the amount is vague consider these three statements in which more information is given.

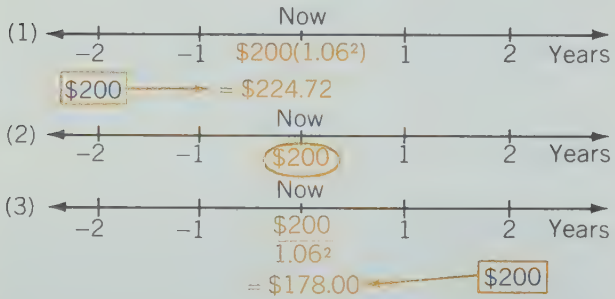
(1) Colleen owes $200; due date: two years ago.

(2) Colleen owes $200; due date: today.

(3) Colleen owes $200; due date: two years from now.

We know that interest must be paid on money owed. Therefore, statement (1) indicates that Colleen owes in excess of $200 – $200 plus some interest. Statement (3) indicates that Colleen's present value indebtedness is less than $200 for she could pay the present value of the $200.

The three situations are shown on time lines. The value today is shown in color. (Interest = 6% compounded annually.)



The value of a sum of money on a certain date is called the *Dated Value*.

EXERCISE 12·11

1. Show on a time line the value today of each dated value.
   (a) $500 payable 3 years ago at 8% compounded semiannually.
   (b) $500 payable today.
   (c) $500 payable 3 years from now at 8% compounded semiannually.
2. Show on a time line the value today of each dated value.
   (a) $1250 payable 5 years from now at 5% compounded annually.
   (b) $1250 payable 5 years ago at 5% compounded annually.
   (c) $1250 payable today.
3. Express in exponential notation the present value of each dated value.
   (a) $326 payable 2 years ago at 5% compounded monthly.
   (b) $756 payable 6 years from now at 10% compounded monthly.
   (c) $3256 payable now.
   (d) $5698 payable $5\frac{1}{2}$ years from now at 8% compounded monthly.
   (e) $725 payable $4\frac{1}{4}$ years ago at 6% compounded quarterly.

12·12 Present Value Using Tables

Working as we did in the previous section we can calculate the present value of $1.00 for 10, 20, 30 and 40 conversion periods for an interest rate of 6% per period. These values are

$i = .06$

n	10	20	30	40
P.V.	0.558	0.312	0.174	0.097

Similarly, $i = .04$

n	10	20	30	40
P.V.	0.676	0.456	0.308	0.208

$i = .02$

n	10	20	30	40
P.V.	0.820	0.673	0.552	0.453

This information is shown on the graph in Figure 12-5. This graph gives us the present value of $1.00 for different n values when $i = 0.02$, 0.04, and 0.06.

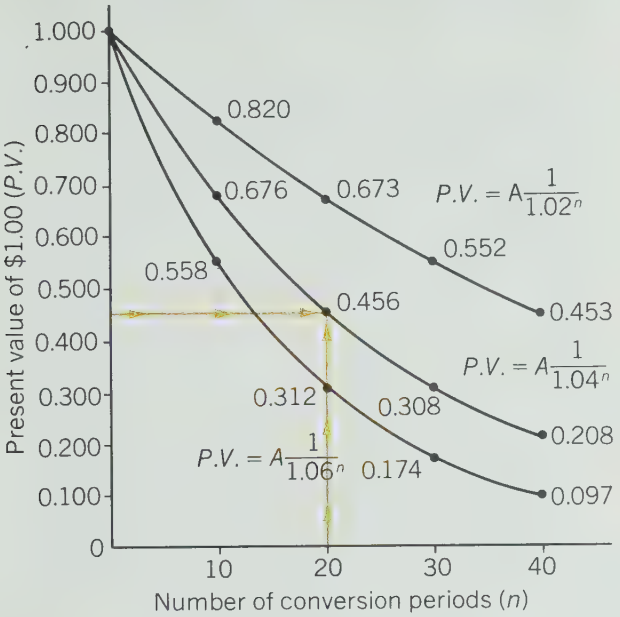


Figure 12-5

Example 1

Using the graph find the present value of $500 due 20 years from now if money is worth 4% compounded annually.

Solution

We use the formula:

$$P.V. = \frac{A}{(1+i)^n}$$

which can be written:

$$P.V. = A \times \frac{1}{(1+i)^n}$$

$$A = \$500$$

$$\frac{1}{1.04^{20}} \simeq 0.46 \qquad \text{(See the colored arrow in Figure 12-5.)}$$

$$P.V. \simeq 500 \times .46$$

$$P.V. \simeq \$230$$

The present value is $230 (approximately).

The graph only provides a rough way to evaluate $\frac{1}{(1+i)^n}$. We can be quite accurate by using the method shown in Section 12·11 but this involved division by numbers with five decimal places: numbers like 3.20714, obtained from the Amount Table. Unless you are using electric calculators such calculations are slow. It is usually much easier and faster to multiply than to divide. Therefore, a table has been made which gives the value of $\frac{1}{(1+i)^n}$. This table is the Present Value Table. It is similar to the Amount Table. An excerpt of the Present Value Table is shown in Table 3.

Table 3

PRESENT VALUE OF 1

Yrs.	3%	3½%	4%	4½%	5%	6%	7%
16	0.62317	0.57671	0.53391	0.49447	0.45811	0.39365	0.33874
17	0.60502	0.55720	0.51337	0.47318	0.43630	0.37136	0.31657
18	0.58739	0.53836	0.49363	0.45280	0.41552	0.35034	0.29586
19	0.57029	0.52016	0.47464	0.43330	0.39573	0.33051	0.27651
20	0.55368	0.50257	0.45639	0.41464	0.37689	0.31181	0.25842

Example 2

Mr. Rodney Sr. wishes to purchase a bond that will pay him $10,000 in 16 years. The value of money is 6% compounded annually. How much will he pay for this bond?

Solution

We will use the Present Value formula:

$$P.V. = \frac{1}{(1+i)^n}$$

We may think of this formula in this form:

$$P.V. = A \times \left[\frac{1}{(1+i)^n}\right]$$

We make these replacements:

$$A = \$10{,}000$$

$$\frac{1}{(1.06)^{16}} = 0.39365 \qquad \text{(from tables)}$$

$$P.V. = 10{,}000\ (0.39365)$$

$$P.V. = 3936.50$$

Mr. Rodney Sr. will pay $3936.50 for the bond.

Example 3

Mr. Goodlad has a debt of $5000 that must be paid 5 years from now. If money is worth 14% per annum compounded quarterly what sum paid now would clear the debt?

Solution

In the formula

$$P.V. = A\left[\frac{1}{(1+i)^n}\right]$$

i is the interest per conversion period and n is the number of conversion periods. Since 14% is the annual interest rate and in this problem the interest is compounded quarterly, $i = \frac{14}{4} = 3\frac{1}{2}\%$ and $n = 5 \times 4 = 20$. Therefore,

$$\begin{aligned} P.V. &= 5000\left[\frac{1}{1.035^{20}}\right] \\ &= 5000\,(0.50257) \qquad \text{(from tables)} \\ &= \$2512.85 \end{aligned}$$

The sum of $2512.85 paid now would clear the debt.

Observe in Example 3 that the table value for $\frac{1}{1.035^{20}}$ is 0.50257 which is approximately one-half. This gives us a quick check: the present value is approximately one-half the amount due. Our answer $2512.83 is roughly one-half of the amount $5000. Present value is *always less* than the amount

EXERCISE 12·12

1. Determine the value of i and n to be used in the Present Value formula: $P.V. = A\left[\frac{1}{(1+i)^n}\right]$.

	Amount	*Due in*	*Annual rate*	*Compounded*
(a)	$5000	5 years	4%	annually
(b)	$600	2 years	6%	annually
(c)	$500	2 years	5%	semiannually
(d)	$300	1 year	9%	monthly
(e)	$10,000	10 years	4%	semiannually
(f)	$4000	5 years	3%	annually

2. Describe in your own words the relation between the present value and time.

3. Describe in your own words the relation between the present value and annual interest rate.

4. Describe in your own words the relation between the time and the annual interest rate if the amount and present value remain constant.

5. Use the graph to find approximate value of $1.00 given:

	Annual rate	*Compounded*	*Time*
(a)	2%	annually	15 years
(b)	4%	annually	25 years
(c)	6%	annually	12 years
(d)	4%	semiannually	8 years

6. Use the Present Value Table to find the value of $1.00 in each part of question 5. Compare your answers to those in question 5.

Use the Present Value Table for the balance of the exercises.

7. Jim Price purchased half interest in a hot dog stand. He still owes $300 payable after 2 years. Since business is so good he decides to pay off the debt now. How much must he pay now if the current rate is 18% compounded monthly?

8. Mark Price won a contest and has some money he wishes to set aside for a motorbike. He is now 14 years old and will buy his bike in 2 years. How much must he invest at 6% compounded semi-annually if the bike cost $640?

9. Janice Page holds two mortgages. The first for $8000 payable 6 years from now contains a clause stating if the mortgage is paid before date due the interest rate to be used to calculate the present value is 5% compounded semiannually. The second for $9000 payable 7 years from now stipulates present value to be calculated on the basis of 6% compounded semiannually. She decides to sell the one that will give her the most money. How much will she receive from the sale?

10. When Donna was born her father purchased a $3500 bond payable on her 21st birthday. Donna who is 17 years old decides to sell the bond so she can go the Technical Institute to study Industrial Laboratory Chemistry. If the bond can be discounted at 10% compounded quarterly, what amount will Donna receive?

11. Donna (question 10) decides to use $1000 and to invest the balance in two parts:

    $1000 at 12% compounded monthly for 1 year. Balance at 12% compounded monthly for 2 years.

    Calculate the interest Donna will receive on her investment.

12. Under what circumstances do you think a student might wisely borrow money to buy (choose any one) a car, a new wardrobe, a motorbike, a typewriter or a skiing outfit.

Using the Library

1. *Romping Through Mathematics* Read pages 8 10 What is $99,999 \times 99,999$?

2. *The Giant Golden Book of Mathematics* Read pages 54 55 Explain the significance of a "Whispering Gallery"

   Summarize the ways various people listed on pages 90 and 91 use mathematics

3. *Mathematics Enrichment* Read pages 138 138 Perform the various exercises Research the library to find what else you can about topology What does the study of topology involve?

REVIEW EXERCISE 12

1. How does the simple interest on a loan vary when only
   (a) the time varies,
   (b) the principal varies,
   (c) the rate varies?

2. How does the rate of simple interest on a loan vary when only
   (a) the interest varies,
   (b) the time varies,
   (c) the principal varies?

3. Without tables or graphs calculate the amount of a loan of $7200 at 12% per annum compounded monthly for 3 months.

4. Use the formula to find the amount of an investment of $1500 at 11% per annum compounded yearly after a time of 3 years.

5. Find the rate of interest per conversion period if the rate is 11% per annum compounded quarterly.

6. Find the number of conversion periods if the term is $9\frac{1}{4}$ years compounded quarterly.

7. Evaluate using tables: $(1.035)^{10}$.

8. Using the Compound Interest Table find the compound amount if $400 is compounded semiannually for 10 years at 8% per annum.

9. Calculate the compound amount of $2255 in 6 years 2 months at 7% per annum compounded semiannually.

10. Determine the value of i and n to be used in the Present Value formula $P.V. = A\left[\frac{1}{(1+i)^n}\right]$ for an amount of $3124 due in 3 years compounded monthly at an annual rate of 6%.

11. Use the graph to find the approximate value of $1 after 12 years invested at 6% per annum compounded 3 times a year.

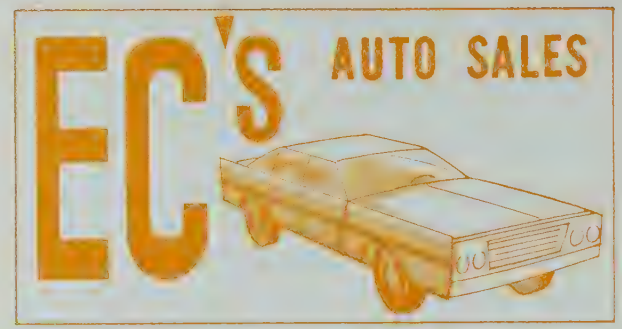
12. Use tables to find the value of $1 in question 11.

13. What message does this ad carry to its readers?

Discuss the situation under which it would

(a) be wise for a person to take a loan to build a pool.

(b) not be wise for a person to take a loan to build a pool.



Today's automobile dealer is in the "Total Transportation" business. One very important and growing facet of this business is "Leasing", and a typical lease rate computation follows this basic pattern.

(1) *Depreciation* – follows closely the Income Tax rate of 30% per year on the reducing balance. On a $4,000 unit, 30% = $1,200, or $100 per month on a twelve month lease.

(2) *Interest* – presently rates are approximately 9.75% per annum, simple interest. Keep in mind that this is also computed on a reducing balance, e.g., as in the example above, the first month you charge on $4,000, the second month on $3,900, and so on. This is then prorated to an equal monthly charge.

(3) The balance of your rental rate is made up of profit required, insurance, maintenance and licence costs – all quoted on a monthly figure, rather than an annual one.

Problem No. 1

Quote a two-year rate on a 1968 Mercury car, valued at cost to the lease department at $4,865, where your money source is 9.75% simple interest. In this example, add $7.50 per month profit. The customer is supplying his own insurance and licence, and agrees to pay maintenance himself. So you need not quote a rate for these.

Problem No. 2

What will the cost be to the lease department at the end of:

A – 12 months?
B – 24 months?

Chapter Thirteen
OBLIQUE TRIANGLES

13·1 Sin θ and Cos θ, $0° \leq \theta \leq 180°$

Let us review the primary trigonometric functions. In a unit circle (radius 1 unit) with centre A the origin, we draw a set of axes and label as in Figure 13-1. We draw a radius $\overline{AC}$ in the first quadrant such that the angle θ is formed. Next, we drop from C a segment perpendicular to the x-axis to meet the x-axis at B. The trigonometric functions are defined by

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

but since $r = 1$ we have for sine and cosine

(1) $\qquad \sin\theta = y \qquad \cos\theta = x.$

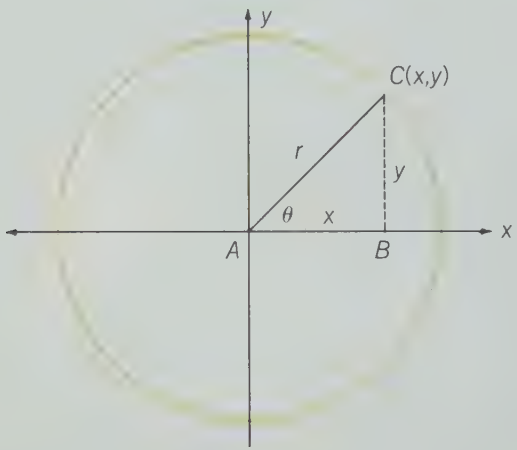


Figure 13-1

Since $\triangle ABC$ (Figure 13-1) is a right triangle we shall use the Pythagorean property to develop an important trigonometric relation.

Notice in Figure 13-1

$$x^2 + y^2 = r^2$$

but $r^2 = 1$ then $x^2 + y^2 = 1$.

From (1) above

$$\cos^2\theta + \sin^2\theta = 1$$

$\cos^2\theta$ means $(\cos\theta)^2$ or $\cos\theta \times \cos\theta$.

Now referring to Figure 13-2 we state the coordinates of C as $(\cos\theta, \sin\theta)$. In this same figure we draw AC' such that $\angle B'AC' = \theta$, and, therefore, $\angle BAC' = 180° - \theta$. The coordinates of C' are $(-x, y)$. We now have

$$\sin\angle BAC' = \frac{y}{r}$$

$$\therefore \quad \sin(180° - \theta) = \frac{y}{r}$$

$$\therefore \quad \sin(180° - \theta) = y \qquad (r = 1)$$

But from (1) $y = \sin\theta$:

(3) $\qquad \sin(180° - \theta) = \sin\theta$

by the transitive property of equality.

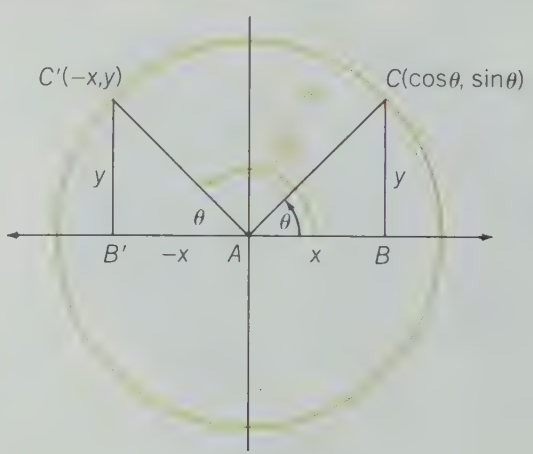


Figure 13-2

We treat $\cos\angle BAC'$ in a similar way.

$$\cos\angle BAC' = \frac{-x}{r}$$

$$\therefore \quad \cos(180° - \theta) = \frac{-x}{r}$$

$$\therefore \quad \cos(180° - \theta) = -x \qquad (r = 1)$$

But from (1) above $x = \cos\theta$

(4) $\therefore$ $\cos(180° - \theta) = -\cos\theta$ *Transitive property*

(5) $\therefore$ $-\cos(180° - \theta) = \cos\theta$

We treat tan $\angle BAC'$ in a similar way.

Show that $\tan(180° - \theta) = -\tan\theta$ (6)

and $\tan\theta = -\tan(180° - \theta)$ (7)

We say $(180° - \theta)$ and θ are supplements since their sum is 180°, i.e., $(180° - \theta) + \theta = 180°$.

The supplement of 160° is 20° since $160° + 20° = 180°$.

Example 1

Express as a ratio of an acute angle

(a) sin 120° (b) cos 130° (c) tan 150°.

Solution

(a) $\sin\theta = \sin(180° - \theta)$ (from (3) above)
$\sin 120° = \sin(180° - 120°)$
$= \sin 60°$

(b) $\cos\theta = -\cos(180° - \theta)$ (from (5) above)
$\cos 130° = -\cos(180° - 130°)$
$= -\cos 50°$

(c) $\tan\theta = -\tan(180° - \theta)$ (from (6) above)
$\tan 150° = -\tan(180° - 150°)$
$= -\tan 30°$

Triangles can be used effectively in architecture.

Example 2

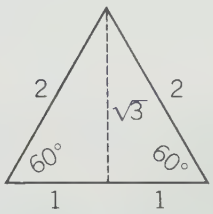
Give the exact value of cos 120°.

Solution

$\cos\theta = -\cos(180° - \theta)$
$\cos 120° = -\cos(180° - 120°)$
$= -\cos 60°$

Now from the triangle we recall

$\cos 60° = \frac{1}{2}$
$\cos 120° = -\cos 60°$
$= -\frac{1}{2}$.



Example 3

Using tables evaluate

(a) sin 112° (b) tan 176°.

Solution

(a) $\sin\theta = \sin(180° - \theta)$
$\sin 112° = \sin(180° - 112°)$
$= \sin 68°$
$\sin 112° = 0.9272$ (*from tables*)

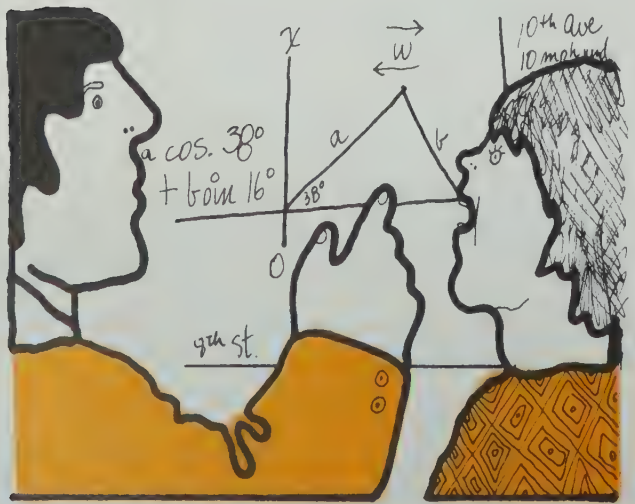
(b) $\tan\theta = -\tan(180° - \theta)$
$\tan 176° = -\tan(180° - 176°)$
$= -\tan 4°$
$= -0.0699$ (*from tables*)

EXERCISE 13·1

1. Write in your own words the rule that covers each statement:
   (a) $\sin 135° = \sin 45°$
   (b) $\cos 127° = -\cos 53°$
   (c) $\tan 105° = -\tan 75°$.

2. Express each as a ratio of an acute angle.
   (a) $\sin 120°$ (b) $\cos 160°$ (c) $\sin 170°$
   (d) $\cos 110°$ (e) $\tan 160°$ (f) $\cos 130°$
   (g) $\tan 127°$ (h) $\sin 156°$ (i) $\cos 138°$
   (j) $\cos 163°$ (k) $\sin 161°$ (l) $\tan 99°$

3. Use tables to evaluate each:
   (a) $\sin 120°$ (b) $\cos 135°$ (c) $\sin 135°$
   (d) $\cos 150°$ (e) $\sin 150°$ (f) $\tan 127°$
   (g) $\tan 142°$ (h) $\tan 161°$ (i) $\sin 111°$
   (j) $\cos 129°$ (k) $\sin 166°$ (l) $\cos 144°$
   (m) $\cos 98°$ (n) $\cos 154°$ (o) $\tan 155°$
   (p) $\tan 133°$ (q) $\sin 173°$ (r) $\tan 91°$.



All I asked was "How far is it to the corner of 8th St. and 10th Ave.?"

13·2 Solution of Oblique Triangles: Sine Law

In the past we have been able to solve algebraically only those triangles which were right triangles. In the section on vectors, Chapter 8, we solved by scale drawings those problems involving triangles which were not right triangles. Triangles which are not right triangles are *oblique triangles*. In this section we will discuss one method of solving oblique triangles and in Section 13·3 we will discuss a second method.

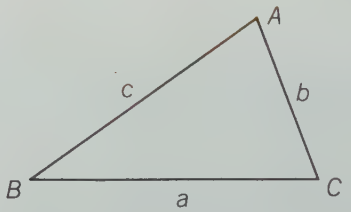
There are six parts to a triangle: three sides and three angles. Knowing just any three does not allow us to solve the triangle. We must know certain combinations of the sides and angles. This chapter will be concerned with two cases only.

Case 1. Given two angles and a side.

To solve a triangle in which we know two angles and a side we introduce a new rule:

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



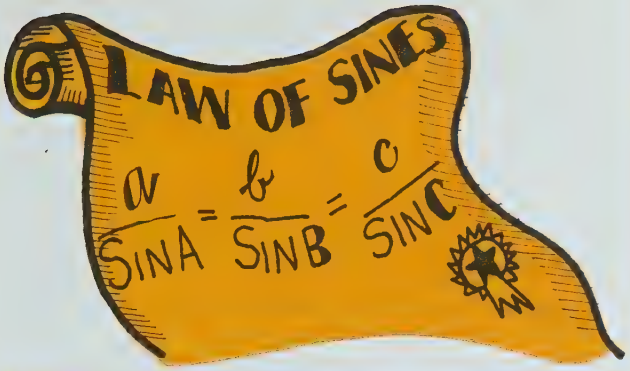
Notice that this Law of Sines shows three separate relations:

(1) $\dfrac{\text{measure of side } a}{\text{sine of angle } A} = \dfrac{\text{measure of side } b}{\text{sine of angle } B}$

(2) $\dfrac{\text{measure of side } b}{\text{sine of angle } B} = \dfrac{\text{measure of side } c}{\text{sine of angle } C}$

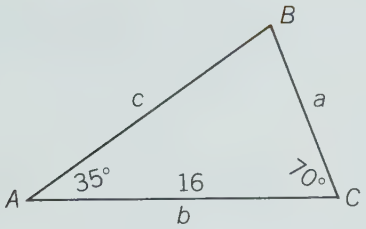
(3) $\dfrac{\text{measure of side } a}{\text{sine of angle } A} = \dfrac{\text{measure of side } c}{\text{sine of angle } C}$

We will now use the Law of Sines to solve a triangle in which we know the measures of two angles and the measure of one side.



Example 1

Solve the triangle below.



Solution

If we know two angles of a triangle we are able to determine the third.

> Sum of the degree measures of the angles of a triangle is 180°.

$$m\angle A + m\angle B + m\angle C = 180°$$
$$m\angle B = 180° - (35° + 70°)$$
$$= 75°$$

We will now use two parts of the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 35°} = \frac{16}{\sin 75°} \qquad \frac{16}{\sin 75°} = \frac{c}{\sin 70°}$$

$$a = \frac{16 \sin 35°}{\sin 75°} \qquad c = \frac{16 \sin 70°}{\sin 75°}$$

$$= \frac{16 \times 0.5736}{0.9659} \qquad = \frac{16 \times 0.9397}{0.9659}$$

$$a = 9.5 \qquad c = 15.5$$

(*slide rule accuracy*)

Project

Project 1

Purpose

To verify the Law of Sines.

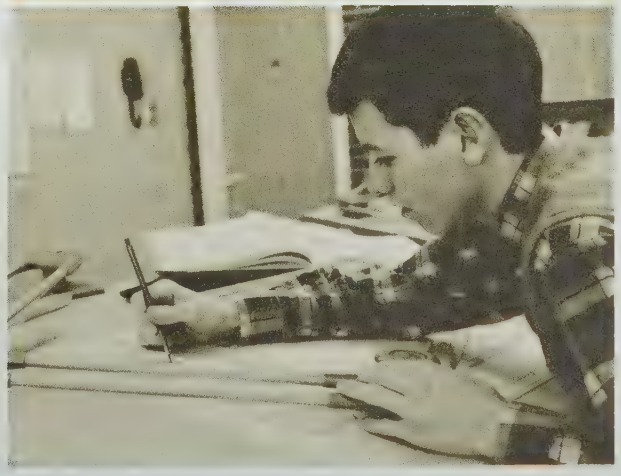
Method

1. Draw three different triangles *ABC*.
2. Measure each side and each angle carefully.
3. Complete this table.

Triangle	1	2	3
a			
$\sin A$			
b			
$\sin B$			
c			
$\sin C$			
$\frac{a}{\sin A}$			
$\frac{b}{\sin B}$			
$\frac{c}{\sin C}$			

Question

1. What can you conclude?



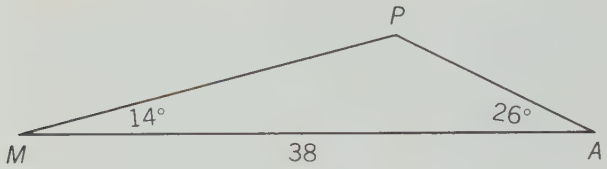
Sometimes we encounter an obtuse angle (that angle whose measure is greater than 90° but less than 180°) in the triangle. We handle these problems as illustrated in Example 1 but with one additional step. Recall, for example,

$$\begin{aligned}\sin 145° &= \sin (180 - 145)° \\ &= \sin 35°\end{aligned}$$

Example 2

Solve $\triangle MAP$ given $m\angle M = 14°$, $m\angle A = 26°$, $p = 38$.

Solution



Step 1. Calculate $m\angle P$.

$$\begin{aligned}m\angle P &= 180° - (14° + 26°) \\ &= 140°\end{aligned}$$

Step 2. Use Law of Sines

$$\frac{p}{\sin P} = \frac{m}{\sin M}$$

$$\frac{38}{\sin 140°} = \frac{m}{\sin 14°}$$

$$\begin{aligned}\sin 140° &= \sin (180 - 140)° \\ &= \sin 40°\end{aligned}$$

$$\frac{38}{\sin 40°} = \frac{m}{\sin 14°}$$

We now proceed to solve for m and subsequently for a. Complete this solution: $m =$ ■ ; $a =$ ■ .

EXERCISE 13·2

1. Write the Law of Sines as it applies to $\triangle MPN$.

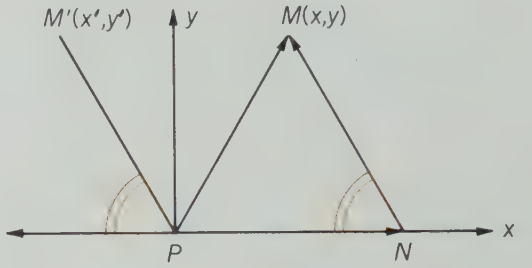
2. Evaluate

   (a) sin 150° (b) sin 130°

   (c) sin 113° (d) sin 167°.

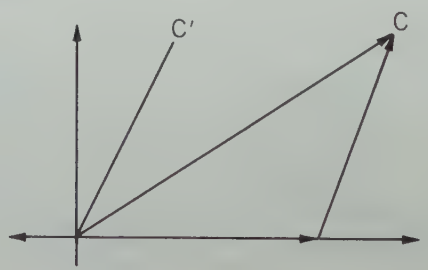
Solve each triangle using the Law of Sines.

3. $\triangle MQR$ given $m\angle M = 32°$, $m\angle Q = 78°$, $m = 23$.
4. $\triangle RST$ given $m\angle R = 43°$, $m\angle S = 88°$, $s = 18$.
5. $\triangle KMN$ given $m\angle K = 126°$, $m\angle N = 11°$, $k = 44$.
6. $\triangle EFG$ given $m\angle G = 128°$, $m\angle F = 24°$, $f = 38.5$.
7. $\triangle CPR$ given $m\angle C = 49°$, $m\angle P = 71°$, $r = 71$.
8. $\triangle RSP$ given $m\angle S = 70°$, $m\angle P = 70°$, $r = 70$.
9. $\triangle MRK$ given $m\angle K = 160°$, $m\angle R = 68°$, $m = 64$.
10. In $\triangle RST$ if $m\angle R = 50°$, $m\angle S = 40°$, find an expression for s in terms of sin 40° and side r. Does this answer look familiar? What special case is this?
11. Find the coordinates of M and M' in terms of sin N and cos N in the diagram below, given $\overline{NM}$ and $\overline{PM'}$.



   (a) Write a vector equation for this question.
   (b) Use the fact that $y' = y$ to derive the Law of Sines.
   (c) Explain why $y' = y$.

12. Write a flow chart for the solution of a triangle given two angles and a side.
13. Develop the Law of Sines using the diagram below.



13·3 Solution of Oblique Triangles: Law of Cosines

So far we have dealt with oblique triangles in which we are given two angles and a side. Now we shall explore the solution of a triangle in which we are given two sides and the *contained angle*: note we said the *contained angle*. By this we mean *the angle formed by the two sides*. Note we cannot solve this case by the Law of Sines. If b, c, and A are given we cannot use

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

since only two of the four quantities are known.

$$\frac{a}{\sin 40^\circ} = \frac{9}{\sin B}$$

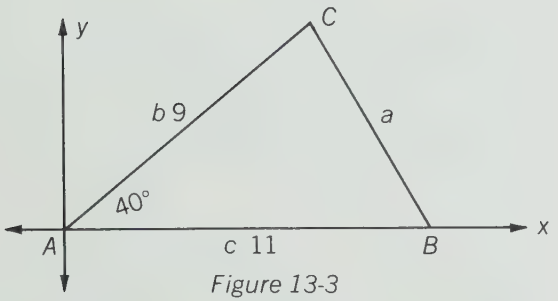


Figure 13-3

Case 2. Given two sides and the contained angle.

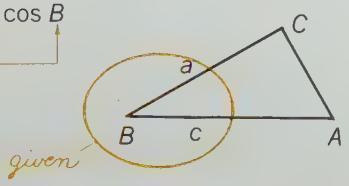
To solve Case 2 problem we use another new rule:

Law of Cosines $\quad a^2 = b^2 + c^2 - 2bc \cos A$

Note how the rule applies to a triangle and the relations within the formula.

While this statement determines a given sides b and c and angle A we can write the Law of Cosines for the triangle given a and c and angle B. To find side b

$b^2 = a^2 + c^2 - 2ac \cos B$



Let us now use the Law of Cosines to solve a triangle when we are given the measures of the two sides and the measure of the contained angle.

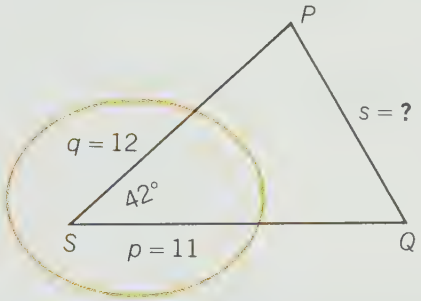
Example 1

Solve $\triangle PQS$ given $p = 11$, $q = 12$ and $m\angle S = 42^\circ$.

Solution

Writing the Law of Cosines in a form suitable for this question we have:

$$s^2 = p^2 + q^2 - 2pq \cos S$$
$$s^2 = 11^2 + 12^2 - 2 \times 11 \times 12 \times \cos 42^\circ$$
$$s^2 = 121 + 144 - 264 \times 0.7431$$
$$s^2 = 68.8$$
$$s = 8.3.$$



Now we can use the Law of Sines to complete the solution. In this case we solve for an angle—not a side as we did in Section 13·2.

$$\frac{s}{\sin S} = \frac{q}{\sin Q}$$
$$\frac{8.3}{\sin 42^\circ} = \frac{12}{\sin Q}$$
$$\sin Q = \frac{12 \times .6691}{8.3}$$
$$\sin Q = 0.9673$$
$$Q = 75^\circ$$

In the table 0.9673 does not occur. The closest number is 0.9659. This is the sine value of 75°. Therefore, 75° is an approximate answer.

To complete the solution we need only calculate the measure of $\angle P$.

$$m\angle P = 180^\circ - (42^\circ + 75^\circ)$$
$$= 63^\circ$$

Project

Project 2

Purpose

To verify the Law of Cosines.

Method

Left to the student. (*Hint!* See Project 1.)

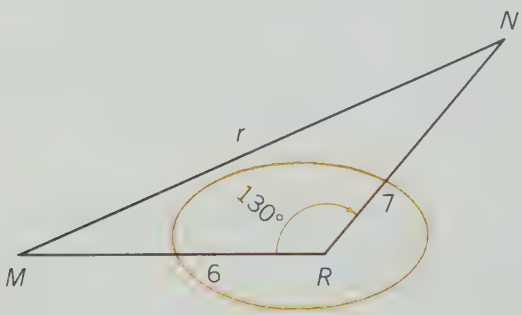
The given angle may be obtuse. Notice in the next example how we handle the obtuse angle.

Example 2

Calculate the measure of $\overline{MN}$ given $m\angle R = 130°$, $m(\overline{RM}) = 6$, $m(\overline{RN}) = 7$.

Solution

Since $\angle R$ is given then we must use $\cos R$ in the Law of Cosines. We find r.



Notice this relation

$$r^2 = m^2 + n^2 - 2mn\cos R$$

$$r^2 = 49 + 36 - 84\cos 130°$$

Now $\cos 130° = \cos(180 - 130)°$

$$= -\cos 50°$$

$\therefore$ $r^2 = 85 - 84(-\cos 50°)$

$r^2 = 85 + 84\cos 50°$ (*Note sign change.*)

$r^2 = 85 + 84 \times 0.6428$

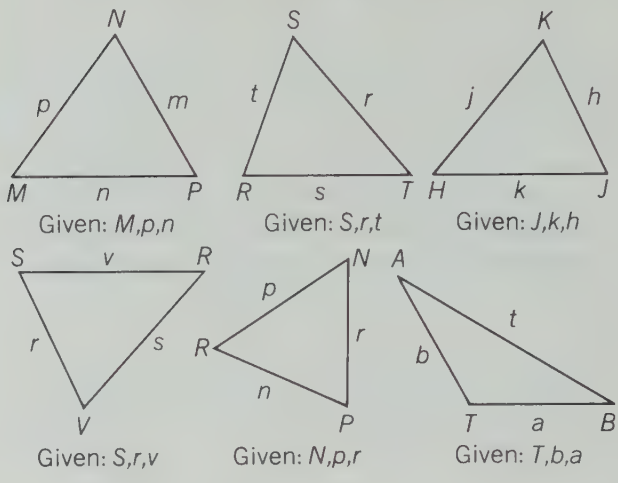
$r^2 = 138$

$r \simeq 11.8$ (*slide rule*)

The length of side NM is 11.8.

EXERCISE 13·3

1. Write the Law of Cosines that would be used in the first step to solve each triangle.



2. In triangle RMB $m\angle R = 32°$, $m(\overline{RM}) = 6$, $m(\overline{RB}) = 5$. Solve the triangle.

3. In triangle TAB $m\angle A = 65°$, $m(\overline{AT}) = 6$, $m(\overline{AB}) = 4$. Solve the triangle.

4. Solve each triangle.

   (a) $m\angle T = 82°$, $m(\overline{PT}) = 8$, $m(\overline{TQ}) = 10$

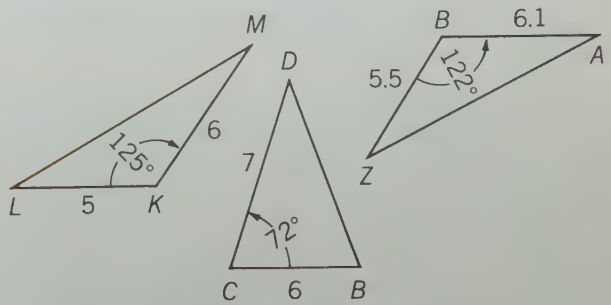
   (b) $m\angle H = 55°$, $m(\overline{HK}) = 12$, $m(\overline{HJ}) = 8$

   (c) $m\angle A = 73°$, $m(\overline{AM}) = 9$, $m(\overline{AR}) = 8$

   (d) $m\angle X = 142°$, $m(\overline{XT}) = 7$, $m(\overline{XR}) = 8$

   (e) $m\angle Z = 132°$, $m(\overline{ZX}) = 15$, $m(\overline{ZY}) = 20$

5. Solve each triangle.



6. Write a flow chart to solve question 2.

13·4 Application

The information we have gained in this chapter can be used to solve problems from other chapters. In particular; we will solve problems relating to the chapters on vectors and trigonometry.

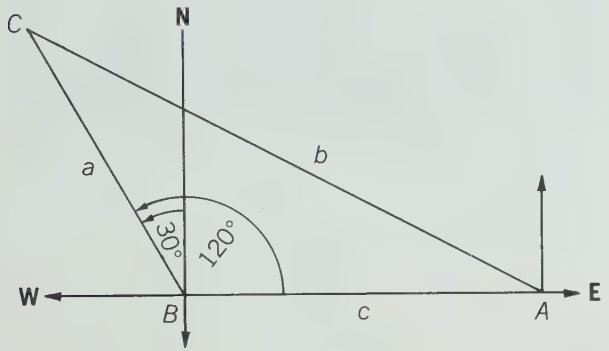
Example 1

A team of surveyors starting from Point A proceeds due west for 10 miles to point B. They then turn and proceed N30°W for 8 miles to point C. Calculate the direct distance and direction point C is from point A.

Solution

In the sections on vectors (8·4, 8·6) you solved questions like this by scale drawings. We will solve this by trigonometry now.

Step 1. Draw the diagram (not necessarily to scale.)



Step 2. Since we know $c = 10$
$a = 8$
$m\angle B = 120°$

We have two sides and the contained angle. Therefore, we use the Cosine Law.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 120° \\ &= 164 + 160\cos(180° - 120°) \\ &= 164 + 160\cos 60° \\ &= 164 + 160 \times 0.5 \end{aligned}$$

$$b^2 = 244$$

$$b = 15.6 \qquad (\textit{slide rule})$$

Step 3. Now use the Sine Law to find the measure of $\angle A$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{8}{\sin A} = \frac{15.6}{\sin 120°}$$

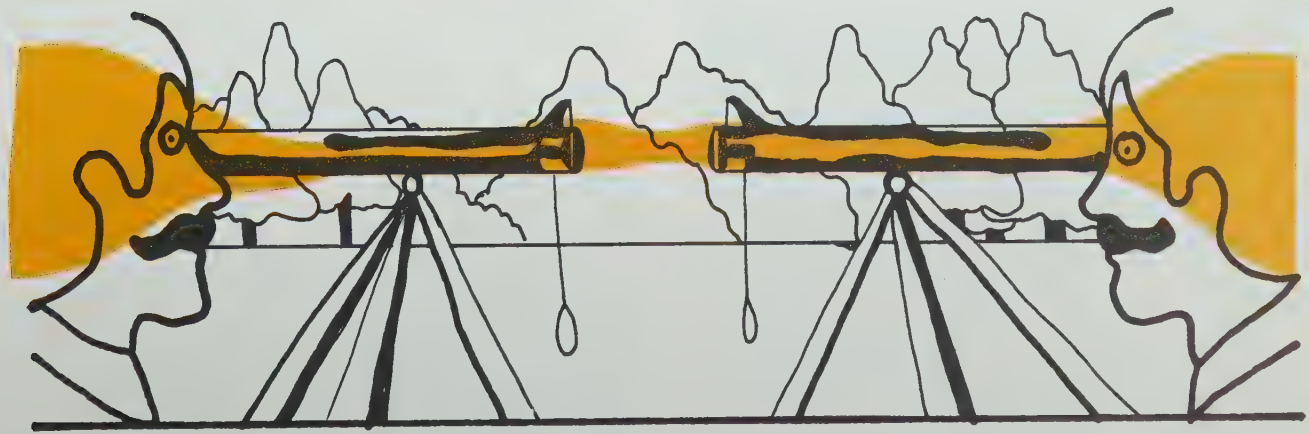
$$[\sin 120° = \sin(180° - 120°)]$$

$$\sin A = \frac{8 \times 0.866}{15.6} \qquad \textit{Did you follow this step?}$$

$$\sin A = 0.444 \qquad (\textit{slide rule})$$

$$A = 26°.$$

The survey crew is 15.6 miles from A at a direction W26°N of A.



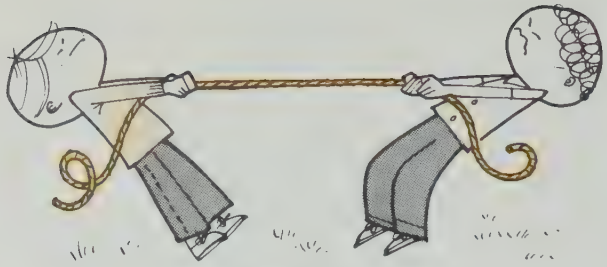


Figure 13-4

Consider two boys pulling in opposite directions on the two ends of a rope. If each boy pulls with the same force then there will be no movement. The two forces will be in *equilibrium*. It is only when one force is greater than the other that the two boys move.

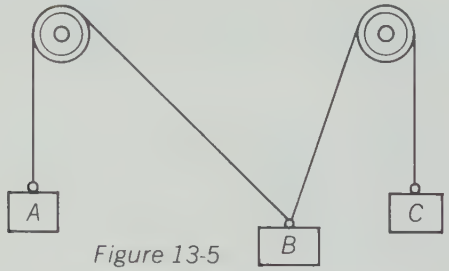


Figure 13-5

Similarly, we experienced the phenomenon of equilibrium in Project 1, page 143, in Chapter 8. When the system was in a state of equilibrium there was no motion. We saw from the project that the vector representing the force and direction of weight B was opposite but equal to the sum of the two vectors representing the forces A and C. In Figure 13-6, $\vec{m}$ represents the force of weight C, $\vec{n}$ the force of weight A and $\vec{r}$ the force of weight B. The sum of $\vec{m}$ and $\vec{n}$ is $\vec{s}$. $\vec{s}$ and $\vec{r}$ are equal in magnitude but opposite in direction when the system is in a state of equilibrium. The diagram in Figure 13-6 is known as the Parallelogram of Forces (8-4).

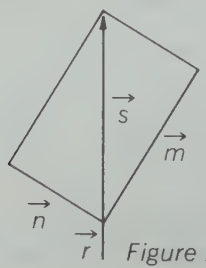
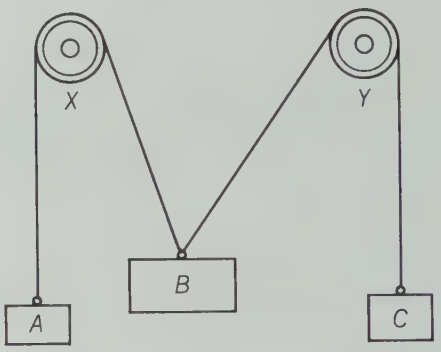


Figure 13-6

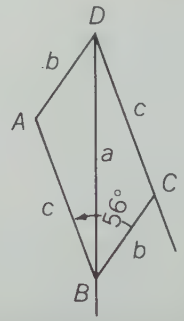
Example 2

The apparatus used in Project 1, page 143, is shown below. The students recorded weights of A and C as 8 pounds and 12 pounds respectively. They forgot to record the weight of B. The diagram showed the measure of $\angle XBY$ as 56°. Find the weight of B.



Solution

Step 1. Draw the parallelogram of forces with its diagonal. We know the diagonal represents the force of weight B. Since $m\angle B = 56°$ then $m\angle A = 180° - 56°$. (Why?) We know the two sides of $\triangle ABD$ and the included angle:
$c = 12, \quad b = 8, \quad m\angle A = 180° - 56°$.



Step 2. We have the two sides and the included angle. We use the Cosine Law

$a^2 = b^2 + c^2 - 2bc \cos A$
$a^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \cos (180° - 56°)$
$a^2 = 208 + 192(0.5592)$ (*Do you follow this?*)
$a^2 = 315.2$
$a = 17.7$ (*slide rule*)

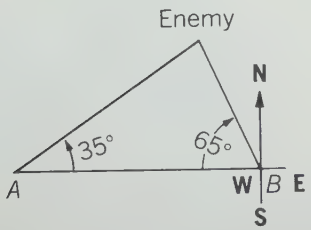
The force due to weight B is 17.7 pounds.

EXERCISE 13·4

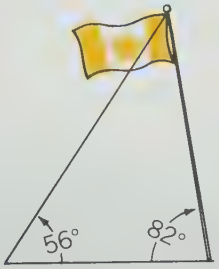
Solve the problems trigonometrically. Use slide rule, and/or calculator and tables to simplify the calculations.

1. A sailboat heads due east for 6 miles, turns and heads N40°E for 10 miles, then stops. Determine the sailboat's final direction and distance from its starting point.

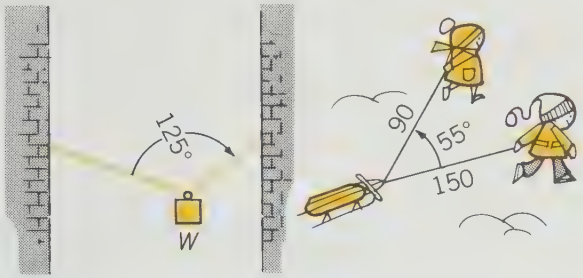
2. Two spotters situated such that A is 4 miles due west of B, spot an enemy gun site. Spotter A sights the enemy E35°N while B spots the enemy W65°N. Determine the distance each spotter is from the enemy gun.



3. A flagpole leans to form an angle of 82°with the horizontal. A trigonometry student lying on the ground 25 feet from the base of the flagpole in the direction the pole leans sights the top of the pole and finds the angle of inclination to be 56°.
   (a) Determine the length of the flagpole.
   (b) (Special.) Find the height of the flagpole.

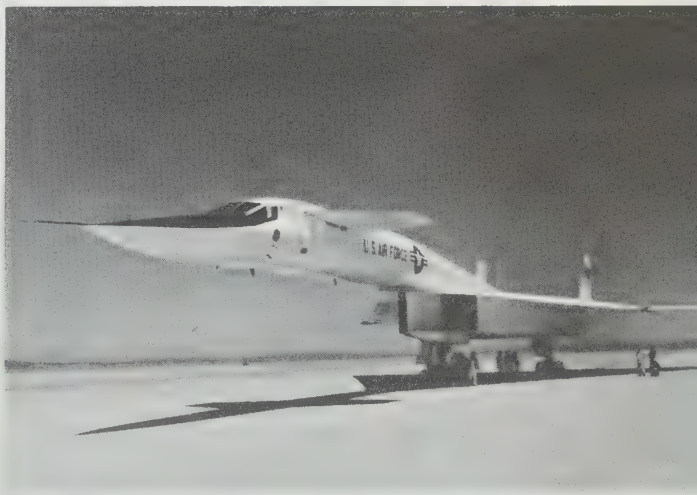


4. A cable is attached to two buildings on opposite sides of a street. A weight is attached to the cable as shown so as to cause the cable to sag. The tension in the long portion of the cable is 125 pounds while the tension in the short portion is 150 pounds. Calculate the weight if the measure of the angle between the two portions is 125°.



5. Two boys pull a sled by means of two ropes which form an angle of 55°. One boy pulls with a force of 150 pounds while the second pulls with a force of 90 pounds. Find the net forward force on the sled.

6. A jet fighter with air speed of 850 miles per hour on course S65°E encounters a jet stream blowing N30°E at 250 miles per hour. Calculate the resultant velocity and track.



7. The pilot of an airplane sets course from Winnipeg N60°E at an air speed of 350 miles per hour. There is a wind blowing W10°N at 90 miles per hour. Calculate the track and ground speed of the aircraft.

Using the Library

1. *The Mathmen.* Read pages 48-50. What is the sum of the angles of a 15-sided polygon?

   Also read pages 77 and 97-98. Explain Eudoxus' theory of the movement of heavenly bodies.

2. *Romping through Mathematics.* Read pages 89-92. Use your hand to estimate the size of buildings, bridges, etc. Practise this method of estimating. What distance must you know?

3. *The Giant Golden Book of Mathematics.* Read pages 88-89. What contributions did Gauss make to astronomy?

REVIEW EXERCISE 13

Express as a trigonometric function of an acute angle.

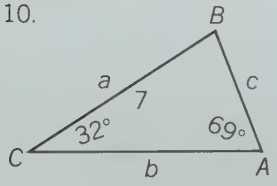
1. sin 135°
2. cos 155°
3. tan 132°.

Use tables to evaluate each.

4. sin 63°
5. cos 72°
6. sin 156°
7. cos 100°
8. tan 42°
9. tan 100°

Write the portion of the Law of Sines you would use in the first step to find the unknown sides in each.

10.



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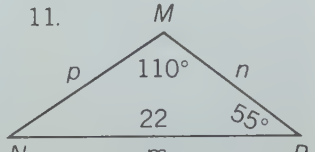
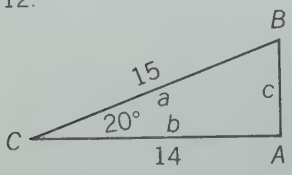


Write the Law of Cosines you would use to solve each.

12.



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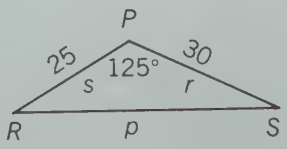


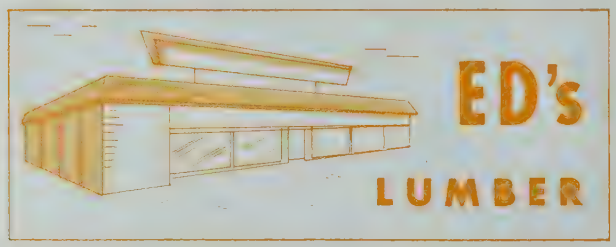
14. Solve the triangle in question 10.

15. Find c in question 12.

16. Calculate the magnitude and direction of the resultant of the following pairs of forces.
   (a) 30 pounds and 50 pounds at an angle of 50° to each other.
   (b) 155 grams and 320 grams at an angle of 120° to each other.

17. Refer to Exercise 8·6. Solve the following questions from this exercise by trigonometry as discussed in this chapter.
   (a) 4 (b) 5 (c) 6

18. Write in words the Law of Cosines.



PROBLEM 1

Customer requests that sufficient 1 × 6 Spruce Shiplap be supplied to complete sheathing the roof on a house under construction. The roof is gable (2 sides only) with 24″ eaves on sides and 18″ overhang on ends. Size of house 24′ wide × 38′ long. Roof pitch 4/12.

PROBLEM 2

How many Board Measure of 3 × 6 Decking would be required to cover a roof area of 1700 sq ft, allowing a covering area of 5″ only due to manufacturing losses and tongue and groove allowance?

PROBLEM 3

A Board Measure of lumber is defined as a piece of material 1″ thick × 12″ long by 12″ wide, or $\frac{1 \times 12}{12} = 1/1 = 1$. Find how many B.M. of lumber is in the following order:

17 Pcs. $1\frac{1}{4}'' \times 19'' - 21'$ Clear Pine

Chapter Fourteen
THE CIRCLE

14-1 Equation of a Circle

In the following project the purposes are set out specifically. The students are urged to search for the relations and they should move as rapidly to the answers as possible; therefore no student need do each question unless difficulty is experienced. Part *A* is given only to establish a pattern of approach to the other parts. This inquiry approach works most efficiently when two or three work together and use the questions only when they cannot proceed by themselves.

Project

Project 1

Purpose

To find the equation (relation) of a circle

(a) with centre at the origin

(b) with centre on the *x*-axis but not at the origin

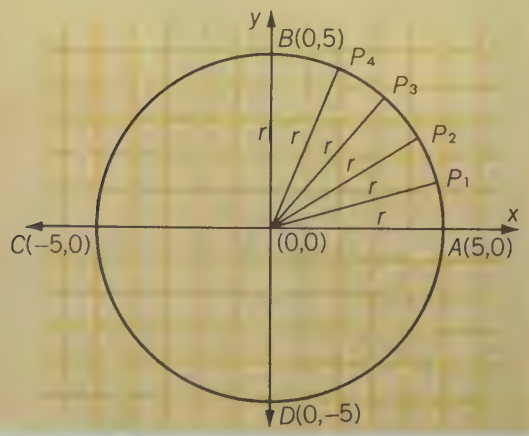
(c) with centre on the *y*-axis but not at the origin

(d) with centre not on either axis.

Part A

Method

1. Draw a circle on graph paper with radius 5 units and centre at the origin, as illustrated.



2. Choose a series of 4 points in the arc *AB*. (Use points with coordinates which are easy to read.)

3. Complete the table.

Point	*x*	*y*	*r*
A	5	0	5
P_1			
P_2			
P_3			
P_4			
B	0	5	5

4. Endeavour to find a relation between *x*, *y* and *r*. *Hint!* Do you recall the distance formula? Remember your results may not be exact since your diagram and the coordinates you found may not be exact.

5. If you got the relation in Step 4 proceed to Step 6. If not, do this step. In Step 3 the value of $x^2 + y^2$ should be the same as r^2. Check to see if this is true for each set of points you picked in Step 3.

   The equation of a circle is $x^2 + y^2 = r^2$ where *r* is the measure of the radius.

6.

Point	*x*	*y*	*r*
P_5			
P_6			
C	−5	0	5
P_7			
P_8			
D			
P_9			
P_{10}			

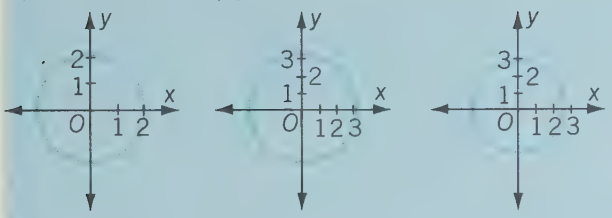
   Choose a series of 2 points in each arc *BC*, *CD* and *DB*. Check the relation you found in Step 4 with these points. Complete the table. Show your test of the relation.

7. What is the equation of the circle with centre at the origin and radius 5 units? 3 units? 4 units?

8. A circle with centre at the origin has a radius of 2 inches. What is the equation of the circle?

9. Give the equation of each circle.

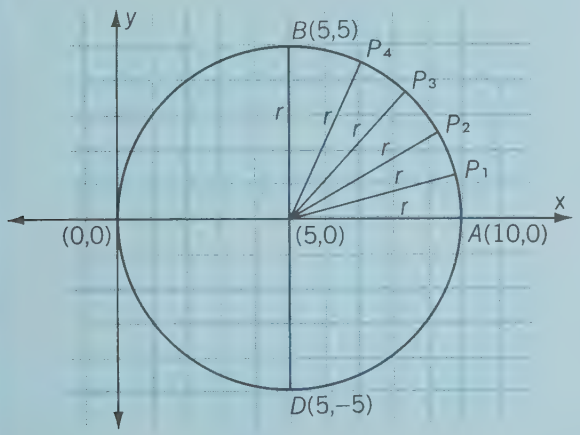
(a) (b) (c)



Part B

Method

1. Draw a circle on graph paper with radius 5 units and centre at (5,0) as illustrated.



2. Choose a series of points in the circle as in Part A.

3. Complete as much of this table as necessary.

Point	x	y	r
A	10	0	5
P_1			
P_2			

4. Can you discover a relation between x, y and r? If not, draw a similar circle to that above but with centre at

(a) (3,0) (b) (–5,0) (c) (–3,0)

and repeat steps 2 and 3.

5. Have you discovered a relation between x, y and r? If you have, go on to 5(e). If not, continue to read this section. The relation is difficult to discover but with a few questions perhaps you will see through the difficulty.

(a) How far to the right of the origin is the centre of the circle?

(b) How far *to the left* would you have to move the circle so the centre would be at the origin?

(c) What number must you subtract from the x-coordinate at the centre to get the x-coordinate at the origin?

(d) In Step 5 of Part *A* you found the relation was

$$x^2 + y^2 = r^2$$

If we substitute the x, y and r values of point *A* in this equation, we get

$$10^2 + 0^2 = 5^2$$

This certainly is not true!

But remember, we said in (c) we had to subtract from the x-coordinates to get the centre of the circle back to the origin. Let us use this as a hint and subtract 5 from the x-coordinate.

$(10 - 5)^2 + 0^2 = 5^2$ Is this true?

Now test another point; and another. Does the relation $(x - 5)^2 + y^2 = r^2$ work when the centre is at (5,0)?

(e) Now we seem to have a relation. Does the relation hold when the centre is at (3,0)? What should this relation be when the centre is at (3,0)? What should the relation be when the centre is at (–5,0)? at (–3,0)?

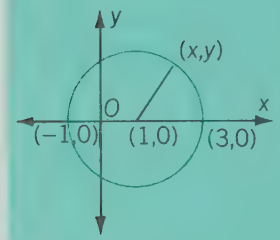
(f) What is the relation of a circle with radius 2 units when the centre is at

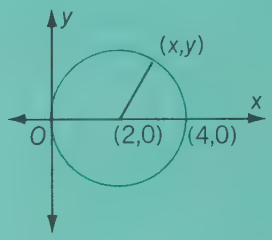
(i) (1,0)? (ii) (3,0)? (iii) (–2,0)?

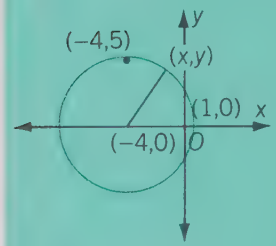
(g) Draw an appropriate circle and choose appropriate points in the circle to test each relation stated in part (f).

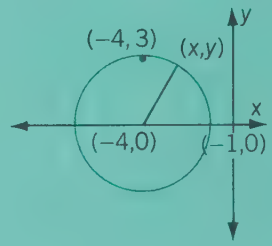
(h) Sketch the circle with the equation
(i) $(x-4)^2+y^2=9$
(ii) $(x+1)^2+y^2=4$
(iii) $(x-2)^2+y^2=1$
(iv) $(x+3)^2+y^2=16$
(v) $(x-a)^2+y^2=r^2 \quad (a>0)$

(i) Give the equation of each circle.





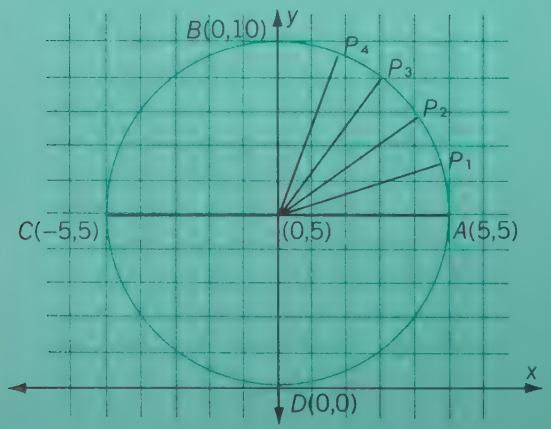




Part C

Method

1. On graph paper draw a circle with radius 5 units and with centre at (0,5) as illustrated.



2. Choose a series of points in the circle as in Part A.

3. Complete as much of this table as necessary.

Point	x	y	r
A	5	5	5
P_1			
P_2			
P_3			

4. Can you discover a relation between x, y and r? If not, draw a similar circle to that above but with centre at
(a) (0,3) (b) (0,−5) (c) (0,−3)
and repeat steps 2 and 3.

5. Have you discovered a relation between x, y and r? If you have, go on to 5(d). If not, continue to read this section. Remember how we reasoned when we moved the circle with centre (0,0) to the right on the x-axis? In this case we moved the circle up on the y-axis.

(a) How far *down* on the y-axis will we have to move the circle to get the centre at the origin?

(b) What must you subtract from the y-coordinate at the centre to get the y-coordinate at the origin?

(c) Then let us subtract 5 from y in the general equation of the circle to get

$$x^2+(y-5)^2=r^2.$$

We can test our relation with $B(0,10)$

$$0^2+(10-5)^2=5^2 \text{ Is this true?}$$

Therefore, it appears the relation is

$$x^2+(y-5)^2=r^2.$$

(d) Test four other points.

(e) Choose two points in each of the other quadrants. Test these points in the new relation.

(f) What do you think is the equation of the circle with radius 4 when the centre is at
(i) (0,1)? (ii) (0,3)?
(iii) (0,−3)? (iv) (0,−5)?

(g) Draw an appropriate circle to test your answer to each part of (f).

(h) Sketch the circle with the equation.

(i) $x^2 + (y - 2)^2 = 4$

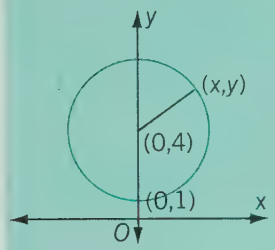
(ii) $x^2 + (y + 3)^2 = 1$

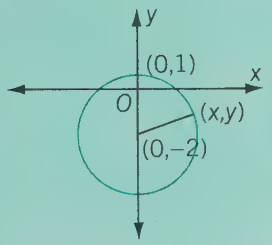
(iii) $x^2 + (y + 1)^2 = 9$

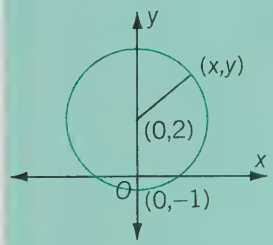
(iv) $x^2 + (y - 2.5)^2 = 6.25$

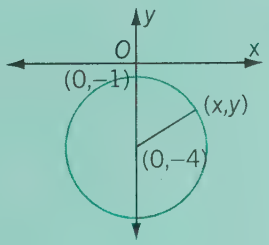
(v) $x^2 + (y - a)^2 = r^2$ $\quad (a > 0)$

(i) State the equation of each circle.





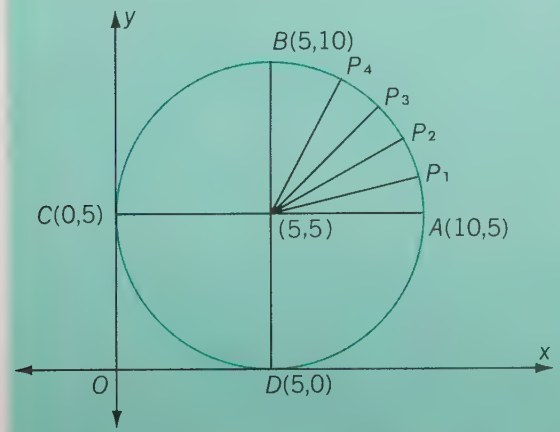




Part D

Method

1. On graph paper draw a circle with radius 5 units and with centre (5,5) as illustrated.



2. Proceed as you have in the previous parts to establish a relation—an equation for the circle.

3. Have you discovered a relation between x, y and r? If you have, go on to 3(d). If not, continue to read this section.

   Recalling what you reasoned in Parts *B* and *C*, can you guess the relation between x, y and r that we should test with the above data? If you cannot guess the relation follow through the questions that follow.

   (a) How did we change the general equation of the circle ($x^2 + y^2 = r^2$) with centre at (0,0) when we moved the circle 5 units to the right? when we moved the circle 5 units up?

   (b) Combining both these changes into one equation we get $(x - \mathbf{?})^2 + (y - \mathbf{?})^2 = r^2$.

   (c) Test the equation in (b) with the data from several points in the arc *CD*. Does the equation check?

   (d) What are the coordinates of the centre of the circle whose equation is

   $$(x - 3)^2 + (y - 2)^2 = 9?$$

   What is the radius?

   (e) What are the coordinates of the centre of each circle if the equation is as shown below? Give the radius of each also?

   (i) $(x - 2)^2 + (y - 7)^2 = 5^2$

   (ii) $(x + 2)^2 + (y + 7)^2 = 6^2$

   (iii) $(x + 3)^2 + (y - 4)^2 = 16^2$

   (f) Write the equation of the circle given the radius and the coordinates of the centre of each.

   (i) $r = 3$, $C(-2,-3)$ $\quad$ (ii) $r = 2$, $C(2,3)$

   (iii) $r = 5$, $C(-4,2)$ $\quad$ (iv) $r = 4$, $C(-3,6)$

(g) Draw the circle with the equation:

(i) $(x-1)^2+(y-2)^2=3^2$

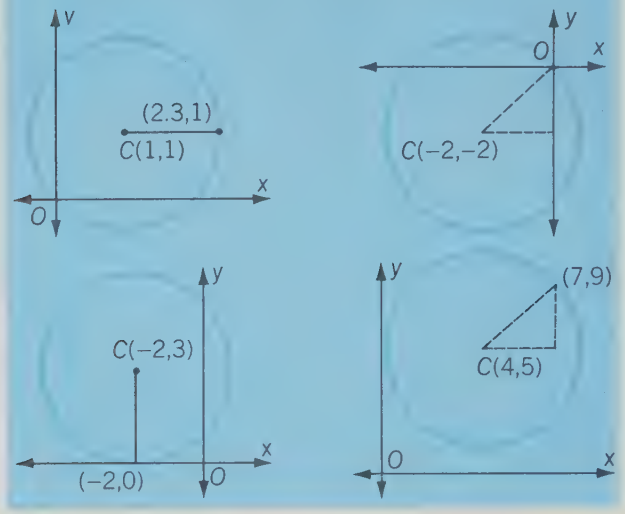
(ii) $(x+2)^2+(y+3)^2=2^2$

(iii) $(x-2)^2+(y+3)^2=9$

(iv) $(x-4)^2+(y-1)^2=25$

(v) $(x+4)^2+(y-2)^2=4$.

(h) Write the equation of each circle.



EXERCISE 14·1

1. We have seen in the preceding project that the equation of a circle is $(x-h)^2+(y-k)^2=r^2$ where h and k are the abscissa and the ordinate of the centre of the circle respectively, and r is the measure of the radius of the circle. This is the standard form of the equation.

   (a) Write the standard form of the equation of a circle when the centre is at the origin.

   (b) Write the standard form of the equation of the circle which has its centre at (3,−4) and has a radius of 6.

2. A circle has the equation $(x-2)^2+(y-3)^2=4^2$. This may be written $(x^2-4x+4)+(y^2-6y+9)=16$. Write the equation of the circle in standard form given its equation $(x^2-10x+25)+(y^2-12y+36)=9$.

3. Write in standard form the equation of the circle whose equation is $(x^2+4x+4)+(y^2-14y+49)=25$.

4. The basic equation $x^2+y^2=9$ is equivalent to $4x^2+4y^2=36$. As in linear equations, equivalent equations have the same solution sets. Therefore the circle $x^2+y^2=9$ is identical to the circle $4x^2+4y^2=36$, $5x^2+5y^2=45$, $6x^2+6y^2=54$, and so on. Write the basic equation of the circle which has an equation of $10x^2+10y^2=160$. What are its centre and its radius?

5. Determine the radius and centre of the circle which has the equation:

   (a) $(x-0)^2+(y-0)^2=4$

   (b) $(x-1)^2+(y-0)^2=9$

   (c) $(x-0)^2+(y+2)^2=8$

   (d) $(x+3)^2+(y+5)^2=8$

   (e) $(x-3)^2+(y+7)^2=12$

   (f) $(x+11)^2+(y-13)^2=169$

   (g) $(x-9)^2+(y-16)^2=143$

   (h) $(x^2+2x+1)+(y^2-2x+1)=5$

   (i) $(x^2+6x+9)+(y^2+10x+25)=12$

   (j) $(x^2+16x+64)+(y^2-14x+49)=24$

   (k) $(x^2+8x+16)+(y^2-12x+36)=36$.

14·2 Interior and Exterior of a Circle

A note about terminology.

We say a point P is in the circle meaning the point P is an element of the set of points satisfying the condition $x^2+y^2=r^2$. (Nonmathematically we might say the point P is on the circumference.)

Points A, B, C are *in the* circle.

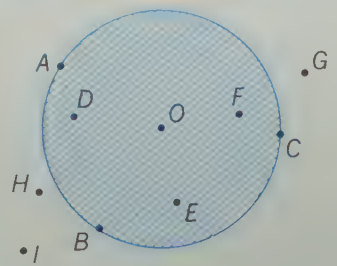


Figure 14-1

We say a point P is *in the interior* of the circle meaning the point P is an element of the set satisfying the condition $x^2 + y^2 < r^2$. (Nonmathematically we might say this point P is inside the circle.)

Points D, E, F are in the interior of the circle.

Finally, we say a point P is *in the exterior* of the circle meaning the point P is an element of the set of points satisfying the condition $x^2 + y^2 > r^2$. (Nonmathematically we might say this point P is outside the circle.)

Points G, H, I are in the exterior of the circle.

Consider the circle in Figure 14-2. Let us determine whether the points $P_1(4,3)$, $P_2(3,3)$, $P_3(4,4)$ are *in the circle*, or *in the interior* of the circle or *in the exterior* of the circle.

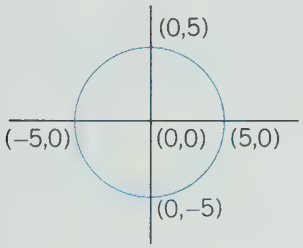


Figure 14-2

We will compare $x^2 + y^2$ and r^2.

For all 3 cases $r^2 = 5^2 = 25$.

For P_1, $x^2 + y^2 = 4^2 + 3^2$
$= 25$

$\therefore$ $x^2 + y^2 = r^2$

Hence, P_1 is *in the circle.*

For P_2, $x^2 + y^2 = 3^2 + 3^2$
$= 18$

$\therefore$ $x^2 + y^2 < r^2$

Hence, P_2 is *in the interior of the circle.*

For P_3, $x^2 + y^2 = 4^2 + 4^2$
$= 32$

$\therefore$ $x^2 + y^2 > r^2$

Hence, P_3 in *in the exterior* of the circle.

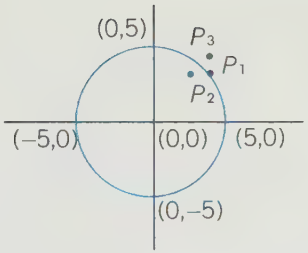


Figure 14-3

EXERCISE 14·2

1. Determine whether or not each point is *in the circle, in the interior* or *in the exterior* of the circle which has the equation $x^2 + y^2 = 10^2$.
   (a) $P_1(4,4)$ (b) $P_2(-3,7)$ (c) $P_3(6,8)$
   (d) $P_4(6,-8)$ (e) $P_5(-6,-9)$ (f) $P_6(7,-6)$

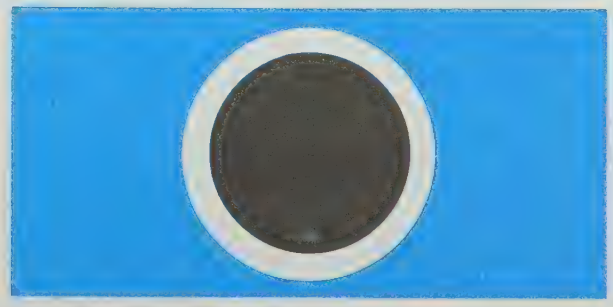
2. Graph each of the following.
   (a) $\{(x,y) | x^2 + y^2 = 36, x, y \in R\}$
   (b) $\{(x,y) | x^2 + y^2 > 36, x, y \in R\}$
   (c) $\{(x,y) | x^2 + y^2 > 81, x, y \in R\}$
   (d) $\{(x,y) | x^2 + y^2 < 49, x, y \in R\}$
   (e) $\{(x,y) | 2x^2 + 2y^2 < 50, x, y \in R\}$

3. Describe the exterior of the circle which has the equation $x^2 + y^2 = 121$ in terms of a set of points.

4. Can a point be in the circle and in the interior of the circle at the same time? Explain.

5. Can a point be in the circle and in the exterior of the circle at the same time? Explain.

6. We have discussed three sets of points with reference to the circle. Use the word *disjoint* and write a sentence about the three sets.



14·3 Properties of a Circle

Each of the following projects is designed to develop a property of the circle. Work each project, study the results and draw a conclusion from your observations. The next section, 14·4, will verify formally some of the properties investigated in this section.

Projects

Purpose of Projects 2 through 7

To develop properties of circles.

Method

Follow the instructions. Study the results. Write a statement of the conclusion reached in each case. It is suggested groups of two or more students should work together with each student individually doing the work, then each in the group looking at each others' finished product. In this way, the student draws a conclusion from two or more instances.

Project 2

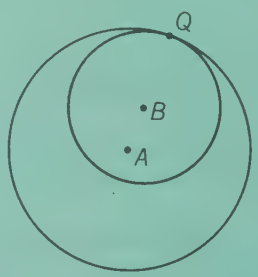
1. Draw three circles of various radii.
2. Draw the diameter $\overline{AB}$ of each.
3. Choose any two points P and P_1, in each circle.
4. Draw $\overline{AP}$, $\overline{PB}$, $\overline{AP_1}$ and $\overline{P_1B}$ in each case.
5. Measure $\angle APB$ and $\angle AP_1B$ in each case.
6. We say $\angle APB$ and $\angle AP_1B$ are *angles in a semicircle.*
7. What is your conclusion about the measure of the *angle in a semicircle?*

Project 3

1. Draw on a sheet of transparent paper (wax paper or onionskin) a circle with radius about 4 inches. Cut out the disc.
2. Fold the disc so that one half of the disc maps onto the other half. Name the crease mark.
3. Repeat question 2 with a second fold. Name the point of intersection of the two crease marks. Conclusion?
4. On the very edge of the disc label any two points A and B. Fold the disc so that A maps onto B. Then open the disc and fold so that A and B are on a fold mark. Name the two fold marks and describe the relations of the two fold marks.
5. On a sheet of paper draw a circle. Label any two points A and B in the circle. Draw $\overline{AB}$. Construct a perpendicular bisector to AB. This perpendicular bisector contains what point of interest? What is your conclusion?

Project 4

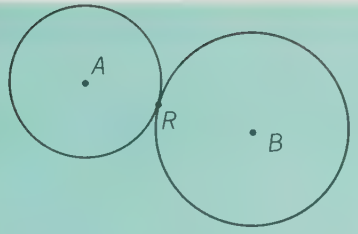
1. Draw a circle with centre A and radius r inches.
2. Draw a second circle with centre B and radius less than r inches in the interior of the first circle such that the two circles intersect in *one* point Q. We say the two circles *intersect internally.*



3. Draw a line to contain points A and B.
4. What do you observe about the line AB and the point Q?
5. What is your conclusion?

Project 5

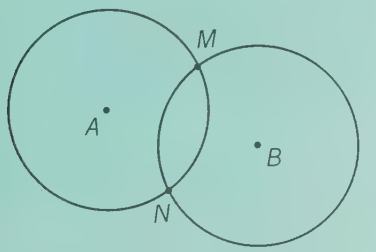
1. Draw a circle with centre A and any radius.
2. Draw a second circle with centre B and any radius in the exterior of the first circle such that the two circles intersect in one point R. We say the two points *intersect externally.*



3. Draw the line AB.
4. What do you observe? What do you conclude?

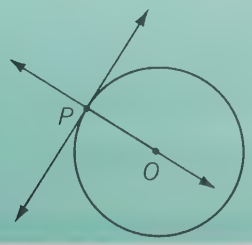
Project 6

1. Draw two circles with centres A and B such that they intersect in two points M and N. Draw $\overleftrightarrow{MN}$ and $\overleftrightarrow{AB}$.
2. What is the relation of the two lines? What is your conclusion?



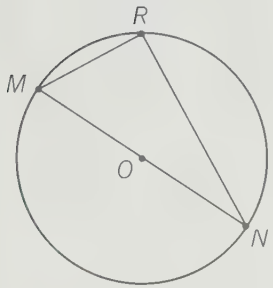
Project 7

1. Draw a circle with radius $\overline{OP}$. Draw $\overleftrightarrow{OP}$.
2. Draw a line that contains one and only one point P in the circle. (This line that *touches* the circle at one point is called the *tangent to the circle*.)
3. What is the relation of the two lines? What is your conclusion?

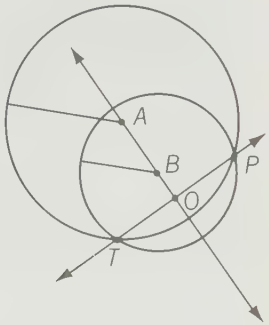


EXERCISE 14·3

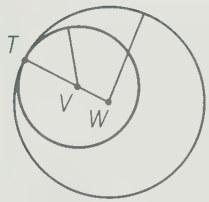
1. Give the measure of each indicated angle. Give a reason for each answer.



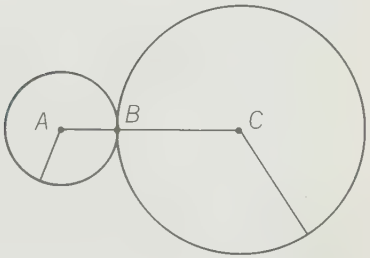
$m\angle MRN = ?$



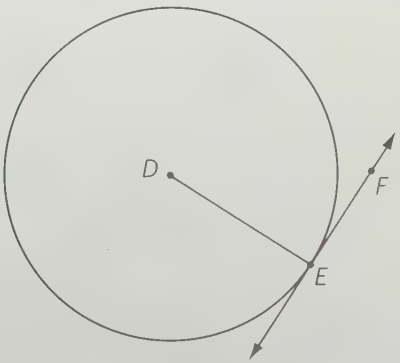
$m\angle AOP = ?$



$m\angle TVW = ?$



$m\angle ABC = ?$



$m\angle DEF = ?$

14·4 Properties of a Circle: Analytical Approach

> The angle formed in a semicircle is a right angle.

In Section 2·4 we learned that two lines are perpendicular if the product of their slopes is -1. We will use this fact to show that the angle formed in a semicircle is a right angle; that is, $\overline{CA}$ is perpendicular to $\overline{CB}$ if slope of $\overline{CA}$ times slope of $\overline{CB}$ is -1. Note $\overline{AB}$ is a diameter of the circle.

To show: (Slope of $\overline{CA}$) $\times$ (Slope of $\overline{CB}$) $= -1$.

Refer now to Figure 14-4. Remember! Slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

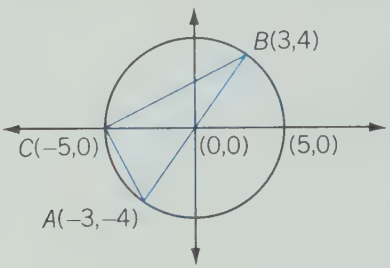


Figure 14-4

$$\text{Slope } m_1 \text{ of } \overline{CA} = \frac{-4-0}{-3-(-5)} = \frac{-4}{2} = -2$$

$$\text{Slope } m_2 \text{ of } \overline{CB} = \frac{4-0}{3-(-5)} = \frac{4}{8} = \frac{1}{2}$$

$$m_1 \times m_2 = -2 \times \tfrac{1}{2}$$

Since (Slope $\overline{CA}$) $\times$ (Slope $\overline{CB}$) $= -1$, therefore, $\overline{CA}$ is perpendicular to $\overline{CB}$.

> The tangent to a circle is perpendicular to that radius which has as an endpoint the point of contact of the tangent.

The circle $(x-2)^2 + (y-3)^2 = 5^2$ has a tangent $\overleftrightarrow{RT}$ at point (6,6). You are given the equation of $\overleftrightarrow{RT}$ is $y = -\frac{4}{3}x + 14$.

We will show that the tangent $\overleftrightarrow{RT}$ is perpendicular to the radius $\overline{RO}$.

$$\text{Slope } m_1 \text{ of } \overleftrightarrow{RT} = -\tfrac{4}{3} \qquad \text{(Why?)}$$

$$\text{Slope } m_2 \text{ of } \overline{OR} = \frac{6-3}{6-2} = \tfrac{3}{4}$$

$$m_1 \times m_2 = -\tfrac{4}{3} \times \tfrac{3}{4} = -1$$

Therefore, since the product of the two slopes is -1 the tangent $\overleftrightarrow{RT}$ is perpendicular to the radius $\overline{OR}$.

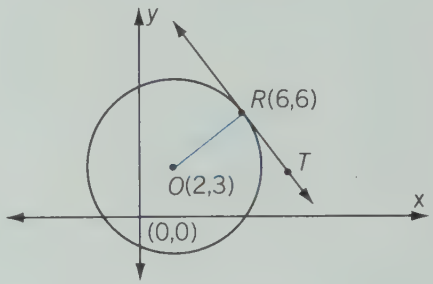


Figure 14-5

Example 1

Find the equation of the tangent at the point (4,5) to the circle $x^2 + y^2 = 41$.

Solution

If we know one point in a line and the slope of the line we can write the equation of that line. (See Chapter 2.)

1. We can find the slope of the radius $\overline{OA}$.
2. We can then find the slope of the tangent.
3. We know one point in the tangent.
4. Therefore, we can write the equation.

Step 1. Slope m_1 of the radius $\overline{OA} = \frac{5}{4}$. *Why?*

Step 2. Slope m_2 of the tangent is $-\frac{4}{5}$ (since $\frac{5}{4} \times -\frac{4}{5} = -1$).

Step 3. Point in tangent is (4,5).

Step 4. $y - y_1 = m(x - x_1)$.

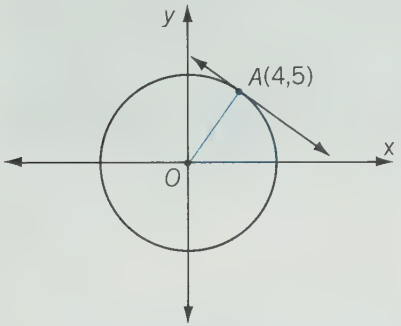
$y - 5 = -\frac{4}{5}(x - 4)$ $\quad (y_1 = 5,\ x_1 = 4)$

$y - 5 = -\frac{4}{5}x + \frac{16}{5}$

$y = -\frac{4}{5}x + \frac{16}{5} + \frac{25}{5}$

$y = -\frac{4}{5}x + \frac{41}{5}$

or $5y + 4x = 41$



When two circles intersect in two points, the line joining the two points is perpendicular to the line containing the centres of the two circles.

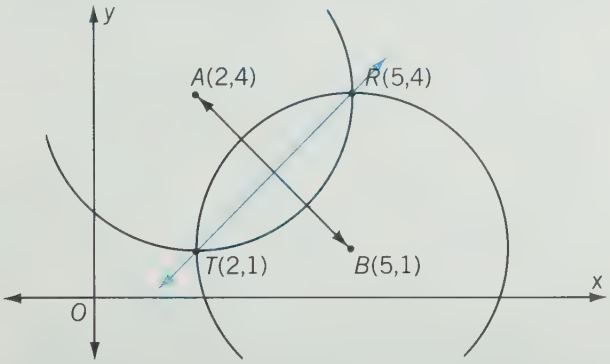


Figure 14-6

We will show that, in Figure 14-6, $\overleftrightarrow{AB} \perp \overleftrightarrow{RT}$.

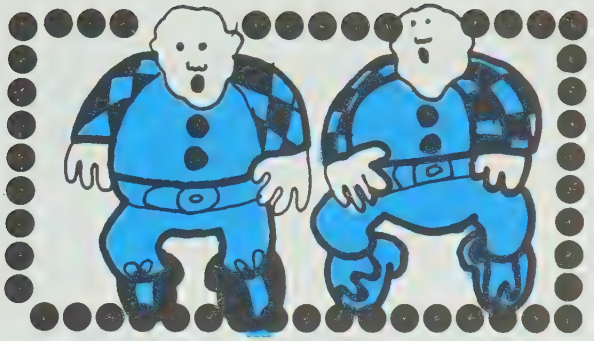
Slope m_1 of $\overleftrightarrow{AB} = -\frac{3}{3} = -1$

Slope m_2 of $\overleftrightarrow{RT} = \frac{3}{3} = 1$

$m_1 \times m_2 = -1 \times 1$

$= -1$

Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{RT}$.



When two circles intersect internally the centres of the circles and the point of intersection are collinear.

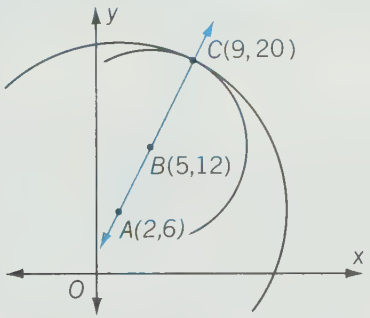


Figure 14-7

Two circles with centres $A(2,6)$ and $B(5,12)$ intersect internally at the point $C(9,20)$. We will show these three points are collinear. We can show this by one of the methods illustrated in Chapter 7. Let us find the slopes of $\overline{AB}$ and $\overline{BC}$. If the two slopes are equal then the three points are collinear.

Slope m_1 of $\overleftrightarrow{AB} = \frac{12 - 6}{5 - 2}$

$= \frac{6}{3}$

$= 2$

Slope m_2 of $\overleftrightarrow{BC} = \frac{20 - 12}{9 - 5}$

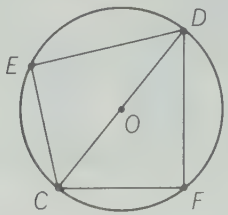
$= \frac{8}{4}$

$= 2$

Since $m_1 = m_2$ and point B is common to both lines, then points A, B, C are collinear. From this example we see that the centres of the two circles that intersect internally and the point of intersection are collinear.

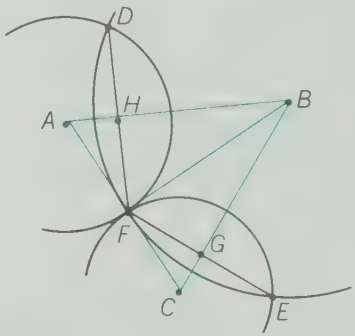
EXERCISE 14·4

1. Given the figure shown below, show that $m\angle DEC = m\angle CFD$. ($O$ is the centre.)



2. The diameter of a circle is $\overline{AB}$: $A(5,0)$, $B(-5,0)$; point C is in the circle. Show analytically that $\angle ACB$ is a right angle.

3. Three circles with centres A, B and C intersect as illustrated. Tell why $\angle DHB \equiv \angle BFG \equiv \angle BGE$.

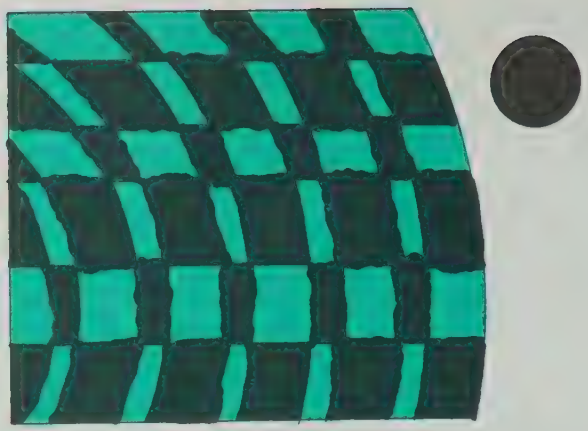


4. Two circles with centres $M(2,4)$ and $N(6,6)$ intersect internally at $P(8,7)$. Show analytically that M, N, and P are collinear.

5. Two circles, $\{(x,y)|x^2 + (y-6)^2 = 36\}$ and $\{(x,y)|(x-6)^2 + y^2 = 36\}$, intersect at $C(0,0)$ and $D(6,6)$. Show analytically that $\overline{AB}$ is perpendicular to $\overline{CD}$, where A and B are the centres.

6. Show in each case that the three points are collinear.

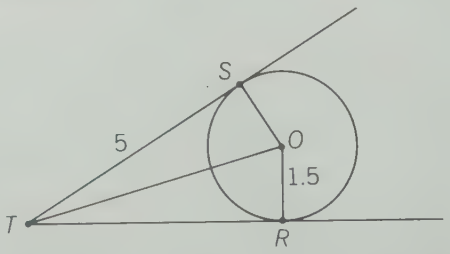
	Centre of		*Point of*
	Circle A	*Circle B*	*intersection Q*
(a)	(2,–4)	(–4,14)	(4,–10)
(b)	(3,–2)	(6,0)	(–3,–6)
(c)	(2,–1)	(4,3)	(–2,–9)
(d)	(1,6)	(6,11)	(3,8)



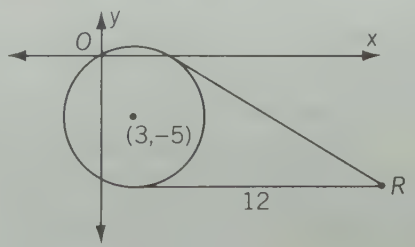
7. Find the equation of the tangent given

	Circle	*Point of contact*
(a)	$\{(x,y)\|x^2 + y^2 = 85\}$	(7,6)
(b)	$\{(x,y)\|x^2 + y^2 = 73\}$	(–3,8)
(c)	$\{(x,y)\|x^2 + (y-3)^2 = 26\}$	(5,4)
(d)	$\{(x,y)\|(x-3)^2 + (y+4)^2 = 101\}$	(4,6)
(e)	$\{(x,y)\|(x+3)^2 + (y+5)^2 = 13\}$	(–5,–2)

8. Find the measure of $\overline{OT}$ in the diagram shown below. Find the measure of $\overline{TR}$. Can you draw any conclusions?



9. A circle $\{(x,y)|(x-3)^2 + (y+5)^2 = 49\}$ has two tangents which intersect at R. The length of each tangent segment is 12. Calculate the distance R is from the centre of the circle.



10. Line AB is tangent to a circle at $B(-5,12)$. The centre of the circle is (0,0). Find the slope of the tangent.

11. Two circles $\{(x,y)|x^2 + y^2 = 25\}$ and $\{(x,y)|x^2 + (y-4)^2 = 9\}$ intersect at $(-3,4)$ and $(3,4)$.

    (a) What is the centre of the first circle? of the second circle?

    (b) What is the slope of the segment joining the two centres?

    (c) What is the slope of the segment joining the two points of intersection of the circles?

    (d) What statement can you make about the two lines in (b) and (c)?

12. Given the diagram with chords $\overline{AB}$ and $\overline{CD}$, $\overline{OM}$ and $\overline{ON}$ represent the distance of $\overline{AB}$ and $\overline{CD}$ respectively from the centre of the circle.

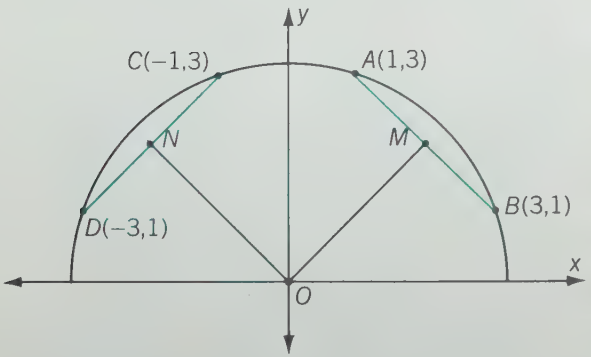
    (a) Calculate $m(\overline{AB})$ and $m(\overline{CD})$.

    (b) Calculate the midpoint of $\overline{AB}$.

    (c) Using the answer from (b) calculate the measure of $\overline{OM}$.

    (d) Calculate $m(\overline{ON})$ in a similar manner.

    (e) What do you notice about the distances two equal chords of a circle are from the centre of the circle? Compare with your classmates.



13. Two chords $\overline{AB}$ and $\overline{CD}$ are drawn in a circle; $m(\overline{AB}) > m(\overline{CD})$. What can you say about the distance of each chord from the centre of the circle?

Using the Library

1. *The Giant Golden Book of Mathematics.* Read pages 77-79. Explain the relation between the note middle C and the note C one octave higher. Make a circular slide rule for finding the scale beginning with any tone.

   Read pages 79-80. Write a sentence about each artist with respect to his contribution to mathematics: Albrecht Dürer, Gentile Bellini.

2. *Mathematics, Enrichment.* Read pages 105-117. Use the information as a starting point, then research and write a paragraph on *Mathematics and Art.*

Albertina Library, Vienna. Reproduced by courtesy of the Medici Society Ltd., London.

Melancholia I. Engraving by Albrecht Dürer, 1514. Is the square magic? What is the shape of that part of the solid figure that you cannot see?

REVIEW EXERCISE 14

1. For each, sketch the circle, give the coordinates of the centre, and state the length of the radius.

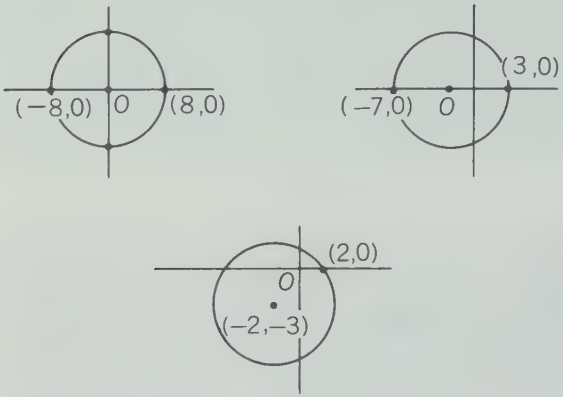
   (a) $x^2 + y^2 = 49$

   (b) $(x + 2)^2 + y^2 = 36$

   (c) $x^2 + (y - 3)^2 = 100$

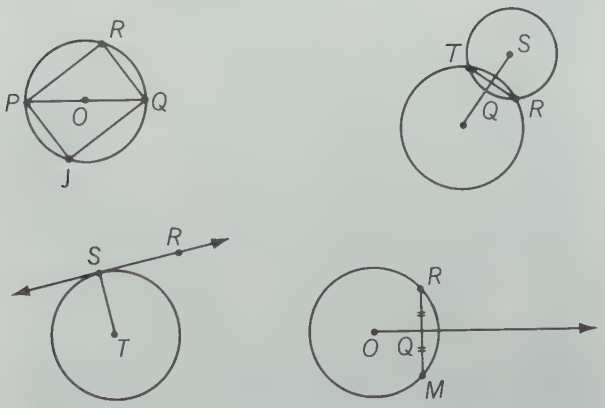
   (d) $(x - 4)^2 + (y + 5)^2 = 81$

2. Write the equation of each circle:



3. Determine where the three points $P_1(2.5, 2.5)$, $P_2(3,2)$, $P_3(2,3)$ lie with respect to the circle $x^2 + y^2 = 3.5^2$.

4. Tell what you know about:



   (a) $\angle PRQ$ and $\angle PJQ$

   (b) $\angle TQS$ and $\angle SQR$

   (c) $\angle RST$

   (d) $\angle RQO$

The Humber Bridge Company fabricates structural steel for a wide variety of projects, such as buildings, bridges, power plants, factories and mines. Most of the contracts are estimated and awarded at a one-lump-sum price. It is extremely important that, firstly, the estimator accurately estimate the amount of material and work content of the project and, secondly, that material and work be utilized efficiently. To ensure that contracts are fabricated for less than the selling price agreed to, internal cost controls are set up to isolate each individual cost so that action can be taken to keep costs to a minimum, and estimates based on the latest available information. It is insufficient to compare total actual cost to total estimated costs – errors may compensate, thus concealing areas in need of improvement.

Problem

Paint is a significant cost on any contract. Cost of paint is dependent on two factors:

(a) The total area to be covered (dependent on the type of material). Estimator must make an assessment of total area.

(b) The coverage, or area covered, per gallon, during application. Coverage is dependent on application thickness. It is waste to over-apply.

Given:

1.	Tonnage of contract	200 tons
2.	Estimated area per ton	175 sq ft/ton
3.	Estimated paint coverage	110 sq ft/gallon
4.	Paint cost per gallon	$12.50/gal
5.	Actual area per ton	150 sq ft/ton
6.	Actual total paint cost	$3900.00

(a) What was the total estimated paint cost?

(b) How did actual total paint cost compare with estimated total paint cost?

(c) What would the company have saved if the contract had been painted at estimated paint coverage (item 3 above)?

POWERS, ROOTS, AND RECIPROCALS 1 – 100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3,420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

TABLE OF SINES, COSINES, TANGENTS

Angle (Degrees)	Sine	Cosine	Tangent
0°	.0000	1.0000	.0000
1°	.0175	.9998	.0175
2°	.0349	.9994	.0349
3°	.0523	.9986	.0524
4°	.0698	.9976	.0699
5°	.0872	.9962	.0875
6°	.1045	.9945	.1051
7°	.1219	.9925	.1228
8°	.1392	.9903	.1405
9°	.1564	.9877	.1584
10°	.1736	.9848	.1763
11°	.1908	.9816	.1944
12°	.2079	.9781	.2126
13°	.2250	.9744	.2309
14°	.2419	.9703	.2493
15°	.2588	.9659	.2679
16°	.2756	.9613	.2867
17°	.2924	.9563	.3057
18°	.3090	.9511	.3249
19°	.3256	.9455	.3443
20°	.3420	.9397	.3640
21°	.3584	.9336	.3839
22°	.3746	.9272	.4040
23°	.3907	.9205	.4245
24°	.4067	.9135	.4452
25°	.4226	.9063	.4663
26°	.4384	.8988	.4877
27°	.4540	.8910	.5095
28°	.4695	.8829	.5317
29°	.4848	.8746	.5543
30°	.5000	.8660	.5774
31°	.5150	.8572	.6009
32°	.5299	.8480	.6249
33°	.5446	.8387	.6494
34°	.5592	.8290	.6745
35°	.5736	.8192	.7002
36°	.5878	.8090	.7265
37°	.6018	.7986	.7536
38°	.6157	.7880	.7813
39°	.6293	.7771	.8098
40°	.6428	.7660	.8391
41°	.6561	.7547	.8693
42°	.6691	.7431	.9004
43°	.6820	.7314	.9325
44°	.6947	.7193	.9657
45°	.7071	.7071	1.0000

Angle (Degrees)	Sine	Cosine	Tangent
45°	.7071	.7071	1.0000
46°	.7193	.6947	1.0355
47°	.7314	.6820	1.0724
48°	.7431	.6691	1.1106
49°	.7547	.6561	1.1504
50°	.7660	.6428	1.1918
51°	.7771	.6293	1.2349
52°	.7880	.6157	1.2799
53°	.7986	.6018	1.3270
54°	.8090	.5878	1.3764
55°	.8192	.5736	1.4281
56°	.8290	.5592	1.4826
57°	.8387	.5446	1.5399
58°	.8480	.5299	1.6003
59°	.8572	.5150	1.6643
60°	.8660	.5000	1.7321
61°	.8746	.4848	1.8040
62°	.8829	.4695	1.8807
63°	.8910	.4540	1.9626
64°	.8988	.4384	2.0503
65°	.9063	.4226	2.1445
66°	.9135	.4067	2.2460
67°	.9205	.3907	2.3559
68°	.9272	.3746	2.4751
69°	.9336	.3584	2.6051
70°	.9397	.3420	2.7475
71°	.9455	.3256	2.9042
72°	.9511	.3090	3.0777
73°	.9563	.2924	3.2709
74°	.9613	.2756	3.4874
75°	.9659	.2588	3.7321
76°	.9703	.2419	4.0108
77°	.9744	.2250	4.3315
78°	.9781	.2079	4.7046
79°	.9816	.1908	5.1446
80°	.9848	.1736	5.6713
81°	.9877	.1564	6.3138
82°	.9903	.1392	7.1154
83°	.9925	.1219	8.1443
84°	.9945	.1045	9.5144
85°	.9962	.0872	11.4301
86°	.9976	.0698	14.3007
87°	.9986	.0523	19.0811
88°	.9994	.0349	28.6363
89°	.9998	.0175	57.2900
90°	1.0000	.0000	undefined

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

COMPOUND AMOUNT OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	1.01000	1.01250	1.01500	1.01750	1.02000	1.02500	1.03000	1.03500	1.04000	1.04500	1.05000	1.06000	1.07000	1.08000	1.09000	1.10000	1
2	1.02010	1.02516	1.03023	1.03531	1.04040	1.05063	1.06090	1.07123	1.08160	1.09203	1.10250	1.12360	1.14490	1.16640	1.18810	1.21000	2
3	1.03030	1.03797	1.04568	1.05342	1.06121	1.07689	1.09273	1.10872	1.12486	1.14117	1.15763	1.19102	1.22504	1.25971	1.29503	1.33100	3
4	1.04060	1.05094	1.06136	1.07185	1.08243	1.10381	1.12551	1.14752	1.16986	1.19252	1.21551	1.26248	1.31080	1.36049	1.41158	1.46410	4
5	1.05101	1.06408	1.07728	1.09062	1.10408	1.13141	1.15927	1.18769	1.21665	1.24618	1.27628	1.33823	1.40255	1.46933	1.53862	1.61051	5
6	1.06152	1.07738	1.09344	1.10970	1.12616	1.15969	1.19405	1.22926	1.26532	1.30226	1.34010	1.41852	1.50073	1.58687	1.67710	1.77156	6
7	1.07214	1.09085	1.10985	1.12912	1.14869	1.18869	1.22987	1.27228	1.31593	1.36086	1.40710	1.50363	1.60578	1.71382	1.82804	1.94872	7
8	1.08286	1.10449	1.12649	1.14888	1.17166	1.21840	1.26677	1.31681	1.36857	1.42210	1.47746	1.59385	1.71819	1.85093	1.99256	2.14359	8
9	1.09369	1.11829	1.14339	1.16899	1.19509	1.24886	1.30477	1.36290	1.42331	1.48610	1.55133	1.68948	1.83846	1.99900	2.17189	2.35795	9
10	1.10462	1.13227	1.16054	1.18944	1.21899	1.28008	1.34392	1.41060	1.48024	1.55297	1.62889	1.79085	1.96715	2.15893	2.36736	2.59374	10
11	1.11567	1.14642	1.17795	1.21026	1.24337	1.31209	1.38423	1.45997	1.53945	1.62285	1.71034	1.89830	2.10485	2.33164	2.58043	2.85312	11
12	1.12683	1.16075	1.19562	1.23144	1.26824	1.34489	1.42576	1.51107	1.60103	1.69588	1.79586	2.01220	2.25219	2.51817	2.81266	3.13843	12
13	1.13809	1.17526	1.21355	1.25299	1.29361	1.37851	1.46853	1.56396	1.66507	1.77220	1.88565	2.13293	2.40985	2.71962	3.06580	3.45227	13
14	1.14947	1.18995	1.23176	1.27492	1.31948	1.41297	1.51259	1.61869	1.73168	1.85194	1.97993	2.26090	2.57853	2.93719	3.34173	3.79750	14
15	1.16097	1.20483	1.25023	1.29723	1.34587	1.44830	1.55797	1.67535	1.80094	1.93528	2.07893	2.39656	2.75903	3.17217	3.64248	4.17725	15
16	1.17258	1.21989	1.26899	1.31993	1.37279	1.48451	1.60471	1.73399	1.87298	2.02237	2.18287	2.54035	2.95126	3.42594	3.97031	4.59497	16
17	1.18430	1.23514	1.28802	1.34303	1.40024	1.52162	1.65285	1.79468	1.94790	2.11338	2.29202	2.69277	3.15882	3.70002	4.32763	5.05447	17
18	1.19615	1.25058	1.30734	1.36653	1.42825	1.55966	1.70243	1.85749	2.02582	2.20848	2.40662	2.85434	3.37993	3.99602	4.71712	5.55992	18
19	1.20811	1.26621	1.32695	1.39044	1.45681	1.59865	1.75351	1.92250	2.10685	2.30786	2.52695	3.02560	3.61653	4.31570	5.14166	6.11591	19
20	1.22019	1.28204	1.34686	1.41478	1.48595	1.63862	1.80611	1.98979	2.19112	2.41171	2.65330	3.20714	3.86968	4.66096	5.60441	6.72750	20
21	1.23239	1.29806	1.36706	1.43954	1.51567	1.67958	1.86029	2.05943	2.27877	2.52024	2.78596	3.39956	4.14056	5.03383	6.10881	7.40025	21
22	1.24472	1.31429	1.38756	1.46473	1.54598	1.72157	1.91610	2.13151	2.36992	2.63365	2.92526	3.60354	4.43040	5.43654	6.65860	8.14027	22
23	1.25716	1.33072	1.40838	1.49036	1.57690	1.76461	1.97359	2.20611	2.46472	2.75217	3.07152	3.81975	4.74053	5.87146	7.25787	8.95430	23
24	1.26973	1.34735	1.42950	1.51644	1.60844	1.80873	2.03279	2.28333	2.56330	2.87601	3.22510	4.04893	5.07237	6.34118	7.91108	9.84973	24
25	1.28243	1.36419	1.45095	1.54298	1.64060	1.85394	2.09378	2.36324	2.66584	3.00543	3.38635	4.29187	5.42743	6.84848	8.62308	10.83471	25
26	1.29526	1.38124	1.47271	1.56998	1.67342	1.90029	2.15659	2.44596	2.77247	3.14068	3.55567	4.54938	5.80735	7.39635	9.39916	11.91818	26
27	1.30821	1.39851	1.49480	1.59746	1.70689	1.94780	2.22129	2.53157	2.88337	3.28201	3.73346	4.82235	6.21387	7.98806	10.24508	13.10999	27
28	1.32129	1.41599	1.51722	1.62541	1.74102	1.99650	2.28793	2.62017	2.99870	3.42970	3.92013	5.11169	6.64884	8.62711	11.16714	14.42099	28
29	1.33450	1.43369	1.53998	1.65386	1.77584	2.04641	2.35657	2.71188	3.11865	3.58404	4.11614	5.41839	7.11426	9.31727	12.17218	15.86309	29
30	1.34785	1.45161	1.56308	1.68280	1.81136	2.09757	2.42726	2.80679	3.24340	3.74532	4.32194	5.74349	7.61226	10.06266	13.26768	17.44940	30
31	1.36133	1.46976	1.58653	1.71225	1.84759	2.15001	2.50008	2.90503	3.37313	3.91386	4.53804	6.08810	8.14511	10.86767	14.46177	19.19434	31
32	1.37494	1.48813	1.61032	1.74221	1.88454	2.20376	2.57508	3.00671	3.50806	4.08998	4.76494	6.45339	8.71527	11.73708	15.76333	21.11378	32
33	1.38869	1.50673	1.63448	1.77270	1.92223	2.25885	2.65234	3.11194	3.64838	4.27403	5.00319	6.84059	9.32534	12.67605	17.18203	23.22515	33
34	1.40258	1.52557	1.65900	1.80372	1.96068	2.31532	2.73191	3.22086	3.79432	4.46636	5.25335	7.25103	9.97811	13.69013	18.72841	25.54767	34
35	1.41660	1.54464	1.68388	1.83529	1.99988	2.37320	2.81386	3.33359	3.94608	4.66734	5.51601	7.68609	10.67658	14.78534	20.41397	28.10244	35
36	1.43077	1.56394	1.70914	1.86741	2.03988	2.43253	2.89827	3.45026	4.10393	4.87737	5.79181	8.14725	11.42394	15.96817	22.25122	30.91268	36
37	1.44508	1.58349	1.73478	1.90009	2.08068	2.49334	2.98522	3.57102	4.26808	5.09686	6.08140	8.63609	12.22362	17.24562	24.25383	34.00395	37
38	1.45953	1.60329	1.76080	1.93334	2.12229	2.55568	3.07478	3.69601	4.43881	5.32621	6.38547	9.15425	13.07927	18.62527	26.43668	37.40434	38
39	1.47412	1.62333	1.78721	1.96717	2.16474	2.61957	3.16702	3.82537	4.61636	5.56589	6.70475	9.70351	13.99482	20.11529	28.81598	41.14478	39
40	1.48886	1.64362	1.81402	2.00160	2.20803	2.68506	3.26203	3.95925	4.80102	5.81636	7.03998	10.28572	14.97446	21.72452	31.40942	45.25926	40

PRESENT VALUE OF 1

Yrs.	1%	1¼%	1½%	1¾%	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%	8%	9%	10%	Yrs.
1	0.99010	0.98765	0.98522	0.98280	0.98039	0.97561	0.97087	0.96618	0.96154	0.95694	0.95238	0.94340	0.93458	0.92593	0.91743	0.90909	1
2	0.98030	0.97546	0.97066	0.96590	0.96117	0.95181	0.94260	0.93351	0.92456	0.91573	0.90703	0.89000	0.87344	0.85734	0.84168	0.82645	2
3	0.97059	0.96342	0.95632	0.94928	0.94232	0.92860	0.91514	0.90194	0.88900	0.87630	0.86384	0.83962	0.81630	0.79383	0.77218	0.75131	3
4	0.96098	0.95152	0.94218	0.93296	0.92385	0.90595	0.88849	0.87144	0.85480	0.83856	0.82270	0.79209	0.76290	0.73503	0.70842	0.68301	4
5	0.95147	0.93978	0.92826	0.91691	0.90573	0.88385	0.86261	0.84197	0.82193	0.80245	0.78353	0.74726	0.71299	0.68058	0.64993	0.62092	5
6	0.94204	0.92817	0.91454	0.90114	0.88797	0.86230	0.83748	0.81350	0.79031	0.76790	0.74622	0.70496	0.66634	0.63017	0.59627	0.56447	6
7	0.93272	0.91672	0.90103	0.88565	0.87056	0.84127	0.81309	0.78599	0.75992	0.73483	0.71068	0.66506	0.62275	0.58349	0.54703	0.51316	7
8	0.92348	0.90540	0.88771	0.87041	0.85349	0.82075	0.78941	0.75941	0.73069	0.70319	0.67684	0.62741	0.58201	0.54027	0.50187	0.46651	8
9	0.91434	0.89422	0.87459	0.85544	0.83675	0.80073	0.76642	0.73373	0.70259	0.67290	0.64461	0.59190	0.54393	0.50025	0.46043	0.42410	9
10	0.90529	0.88318	0.86167	0.84073	0.82035	0.78120	0.74409	0.70892	0.67556	0.64393	0.61391	0.55840	0.50835	0.46319	0.42241	0.38554	10
11	0.89632	0.87228	0.84893	0.82627	0.80426	0.76214	0.72242	0.68495	0.64958	0.61620	0.58468	0.52679	0.47509	0.42888	0.38753	0.35049	11
12	0.88745	0.86151	0.83639	0.81206	0.78849	0.74356	0.70138	0.66178	0.62460	0.58966	0.55684	0.49697	0.44401	0.39711	0.35553	0.31863	12
13	0.87866	0.85087	0.82403	0.79809	0.77303	0.72542	0.68095	0.63940	0.60057	0.56427	0.53032	0.46884	0.41496	0.36770	0.32618	0.28966	13
14	0.86996	0.84037	0.81185	0.78436	0.75788	0.70773	0.66112	0.61778	0.57748	0.53997	0.50507	0.44230	0.38782	0.34046	0.29925	0.26333	14
15	0.86135	0.82999	0.79985	0.77087	0.74302	0.69047	0.64186	0.59869	0.55526	0.51672	0.48102	0.41727	0.36245	0.31524	0.27454	0.23939	15
16	0.85282	0.81975	0.78803	0.75762	0.72845	0.67363	0.62317	0.57671	0.53391	0.49447	0.45811	0.39365	0.33874	0.29189	0.25187	0.21763	16
17	0.84438	0.80963	0.77639	0.74459	0.71416	0.65720	0.60502	0.55720	0.51337	0.47318	0.43630	0.37136	0.31657	0.27027	0.23107	0.19784	17
18	0.83602	0.79963	0.76491	0.73178	0.70016	0.64117	0.58739	0.53836	0.49363	0.45280	0.41552	0.35034	0.29586	0.25025	0.21199	0.17986	18
19	0.82774	0.78976	0.75361	0.71919	0.68643	0.62553	0.57029	0.52016	0.47464	0.43330	0.39573	0.33051	0.27651	0.23171	0.19449	0.16351	19
20	0.81954	0.78001	0.74247	0.70682	0.67297	0.61027	0.55368	0.50257	0.45639	0.41464	0.37689	0.31181	0.25842	0.21455	0.17843	0.14864	20
21	0.81143	0.77038	0.73150	0.69467	0.65978	0.59539	0.53755	0.48557	0.43883	0.39679	0.35894	0.29416	0.24151	0.19866	0.16370	0.13513	21
22	0.80340	0.76087	0.72069	0.68272	0.64684	0.58087	0.52189	0.46915	0.42196	0.37970	0.34185	0.27751	0.22571	0.18394	0.15018	0.12285	22
23	0.79544	0.75147	0.71004	0.67098	0.63416	0.56670	0.50669	0.45329	0.40573	0.36335	0.32557	0.26180	0.21095	0.17032	0.13778	0.11168	23
24	0.78757	0.74220	0.69954	0.65944	0.62172	0.55288	0.49193	0.43796	0.39012	0.34770	0.31007	0.24698	0.19715	0.15770	0.12640	0.10153	24
25	0.77977	0.73303	0.68921	0.64810	0.60953	0.53939	0.47761	0.42315	0.37512	0.33273	0.29530	0.23300	0.18425	0.14602	0.11597	0.09230	25
26	0.77205	0.72398	0.67902	0.63695	0.59758	0.52624	0.46369	0.40884	0.36069	0.31840	0.28124	0.21981	0.17220	0.13520	0.10639	0.08390	26
27	0.76440	0.71505	0.66899	0.62599	0.58586	0.51340	0.45019	0.39501	0.34682	0.30469	0.26785	0.20737	0.16093	0.12519	0.09761	0.07628	27
28	0.75684	0.70622	0.65910	0.61523	0.57438	0.50088	0.43708	0.38165	0.33348	0.29157	0.25509	0.19563	0.15040	0.11591	0.08955	0.06934	28
29	0.74934	0.69750	0.64936	0.60465	0.56311	0.48866	0.42435	0.36875	0.32065	0.27902	0.24295	0.18456	0.14056	0.10733	0.08215	0.06304	29
30	0.74192	0.68889	0.63976	0.59425	0.55207	0.47674	0.41199	0.35628	0.30832	0.26700	0.23138	0.17411	0.13137	0.09938	0.07537	0.05731	30
31	0.73458	0.68004	0.63031	0.58403	0.54125	0.46512	0.39999	0.34423	0.29646	0.25550	0.22036	0.16426	0.12277	0.09202	0.06915	0.05210	31
32	0.72730	0.67198	0.62099	0.57398	0.53063	0.45377	0.38834	0.33259	0.28506	0.24450	0.20987	0.15496	0.11474	0.08520	0.06344	0.04736	32
33	0.72010	0.66369	0.61182	0.56411	0.52023	0.44270	0.37703	0.32134	0.27409	0.23397	0.19987	0.14619	0.10724	0.07889	0.05820	0.04306	33
34	0.71297	0.65549	0.60277	0.55441	0.51003	0.43191	0.36605	0.31048	0.26355	0.22390	0.19036	0.13791	0.10022	0.07305	0.05339	0.03914	34
35	0.70591	0.64740	0.59387	0.54487	0.50003	0.42137	0.35538	0.29998	0.25341	0.21425	0.18129	0.13011	0.09366	0.06763	0.04899	0.03558	35
36	0.69893	0.63941	0.58509	0.53550	0.49022	0.41109	0.34503	0.28983	0.24367	0.20503	0.17266	0.12274	0.08754	0.06263	0.04494	0.03235	36
37	0.69201	0.63151	0.57644	0.52629	0.48061	0.40107	0.33498	0.28003	0.23430	0.19620	0.16444	0.11579	0.08181	0.05799	0.04123	0.02941	37
38	0.68515	0.62372	0.56792	0.51724	0.47119	0.39129	0.32523	0.27057	0.22529	0.18775	0.15661	0.10924	0.07646	0.05369	0.03783	0.02673	38
39	0.67837	0.61602	0.55953	0.50834	0.46195	0.38174	0.31575	0.26141	0.21662	0.17967	0.14915	0.10306	0.07146	0.04971	0.03470	0.02430	39
40	0.67165	0.60841	0.55126	0.49960	0.45289	0.37243	0.30656	0.25257	0.20829	0.17193	0.14205	0.09722	0.06678	0.04603	0.03184	0.02209	40

Answers

CHAPTER 1

EXERCISE 1·3, Page 6

1. (a) direct (b) partial (c) direct (d) direct (e) direct
(f) direct (g) direct (h) direct (i) partial (j) direct (k) partial
3. (d) 100 minutes (e) 0.35 ft, 2 ft (f) 0.10 ft, constant up to $t = 100$ min
(g) yes for $t \leq 50$, yes for $t \leq 33\frac{1}{3}$, yes for $t \leq \frac{100}{n}$ (h) yes (i) direct

EXERCISE 1·4, Page 8

2. (b) 6.65 ft, 5 ft (c) 70 min (d) 256 cu ft, 193 cu ft
(e) 0.10 ft, yes for $t \leq 70$ (f) no, no (g) yes (h) partial
3. (b) linear on each interval: $0 \leq t \leq 2$, $2 \leq t \leq 4$. (d) no
(e) direct on first interval, partial on second. **4.** direct

EXERCISE 1·5, Page 10

1. 125 sec **2.** 1.9 sec **3.** increases **4.** decreases **5.** 102 sec

EXERCISE 1·6, Page 12

1. (a) 9.7 cm (b) 10.55 cm (c) 11.1 cm **2.** (a) 13.9 cm (b) 14.4 cm
(c) 15.7 cm (d) 8.2 cm **3.** (a) 29.7 cm (b) 28.9 cm (c) 22.2 cm (d) 33.6

EXERCISE 1·7, Pages 13-14

7. (b) (i) 13.0 (ii) 28.5 (iii) −0.7 (iv) 41.3

EXERCISE 1·8, Pages 15-16

1. (a) 200′ (b) 364′ (c) 544′ **2.** (a) 200′ (b) 360′ (c) 575′ **3.** (a) 1.8 sec
(b) $2\frac{1}{2}$ sec (c) 3.1 sec (d) 3.5 sec **4.** (d) 400′ (e) 4.3 sec (f) 6.1 sec **5.** (b) 45.0 mi,
48.3 mi, 51.6 mi (c) 40 mi E, 12 mi N, 41.8 mi; 40 mi E, 20 mi N, 44.7; 40 mi E, 28 mi N, 48.8 mi
(d) 40.4 mi, 40.8 mi, 43.0 mi, 46.6 mi, 51.2 mi, 54.7 mi (e) direct, partial (f) no, no, not linear.
6. (a) −4.25 (b) no **7.**(a) (i) 510°F (ii) 370°F (iii) 12,000°F (b) 1800 mph
(c) 5800 mph **8.** (c) (i) 118 (ii) 123 (iii) 158 (d) yes (e) partial

REVIEW EXERCISE 1, Pages 19-20

1. nonlinear function; direct variation, linear function; partial variation, linear function; partial variation, linear function; nonfunction; nonfunction **2.** (a) (i) 4.8 (ii) 15.6

CHAPTER 2

EXERCISE 2·1, Pages 23-24

1. no, 5 is mapped onto more than one value of y. **2.** yes, just one y-value for each x-value.
6. (a) $y = (-1)x = 8$ (b) $y = x + 5$ (c) $y = -\frac{2}{3}x + 6$ (d) $y = -\frac{2}{3}x + (-3)$
(e) $y = \frac{1}{2}x + (-2)$ (f) $y = 3x + 12$ (g) $y = \frac{2}{5}x + (-\frac{8}{5})$ (h) $y = -2x + \frac{3}{7}$ (i) $y = \frac{1}{2}x + (-2)$
(j) $y = \frac{3}{4}x + 0$ (k) $y = \frac{1}{3}x + (-2)$ (l) $y = 2x + 0$ (m) $y = -x + 0$ (n) $y = -\frac{1}{2}x + 0$
(o) $y = \frac{8}{9}x + 0$ (p) $y = x + (-4)$ **7.** (a) $-40 \leq C \leq 40$, $C \in R$ (c) $-40 \leq F \leq 104$, $F \in R$

EXERCISE 2·2, Pages 25-26

1. (a) $y = \frac{2}{3}$ (b) $y = \frac{6}{5}$ (c) $y = -\frac{5}{4}$ (d) $y = -\frac{C}{B}$ **2.** (a) $x = -\frac{2}{5}$ (b) $x = \frac{5}{6}$
(c) $x = -\frac{2}{3}$ (d) $x = -\frac{C}{A}$ **3.** (a) $\frac{2}{3}$ (b) $\frac{6}{5}$ (c) $-\frac{5}{4}$ (d) $-\frac{C}{B}$
4. (a) $-\frac{2}{5}$ (b) $\frac{5}{6}$ (c) $-\frac{2}{3}$ (d) $-\frac{C}{A}$
5. (a) $0x + 1y + 3 = 0$ (b) $1x + 0y - 8 = 0$ (c) $1x - 3y + 2 = 0$ (d) $2x - 1y + 4 = 0$
(e) $0x + 1y - 3 = 0$ (f) $1x + 0y + \frac{2}{3} = 0$ (g) $1x + 4y - 3 = 0$ (h) $3x - 2y - \frac{2}{3} = 0$
6. (a) $y = -\frac{3}{2}x - \frac{5}{2}$ (b) $y = \frac{2}{3}x - 7$ (c) $y = \frac{12}{7}x + \frac{12}{7}$ (d) $y = -2x + \frac{49}{48}$
(e) $y = \frac{1}{9}x - 48$ (f) $y = 15x - \frac{35}{2}$ **9.** (a) $S = 2T + 1089$ (b) $-40 \leq T \leq 100$, $T \in R$
(c) $1009 \leq S \leq 1289$ (d) yes, only one S for each T

EXERCISE 2·3, Pages 28-29

1. (a) $\frac{3}{4}$ (b) $-\frac{37}{35}$ (c) $-\frac{8}{17}$ (d) $\frac{46}{81}$ (e) $\frac{7}{6}$ (f) $-\frac{7}{15}$
(g) $\frac{y-6}{x-8}$ (h) $\frac{y-3}{x-4}$ (i) $\frac{y-2}{x+6}$ (j) 0 (k) undefined (l) $\frac{y-y'}{x-x'}$
2. $y = \frac{5}{2}x - \frac{11}{2}$ **3.** $y = -\frac{4}{3}x - \frac{14}{3}$ **4.** $y = \frac{5}{3}x - 3$ **5.** $y = -\frac{4}{3}x - \frac{2}{3}$
6. By measurement, slope of each $= \frac{1}{2}$. (g) $y = \frac{1}{2}x + 2$ (h) $y = \frac{1}{2}x + 1$ (i) $y = \frac{1}{2}x + 5$
7. (a) $y = \frac{3}{4}x + \frac{5}{2}$ (b) $y = -\frac{37}{35}x + \frac{601}{70}$ (c) $y = -\frac{8}{17}x + \frac{90}{17}$ (d) $y = \frac{46}{81}x + \frac{77}{27}$ (e) $y = \frac{7}{6}x + \frac{3}{2}$
(f) $y = -\frac{7}{15}x - \frac{18}{5}$ **8.** (j) $y = 3$ (k) $x = 3$ (not a function) (l) Slope $= 1$, $y = x - x' + y'$

EXERCISE 2·4, Page 30

1. 2 **2.** $-\frac{1}{2}$ **3.** (a) (i) 5 (ii) $-\frac{1}{5}$ (b) (i) $-\frac{3}{2}$ (ii) $\frac{2}{3}$
(c) (i) -2 (ii) $\frac{1}{2}$ (d) (i) $-\frac{4}{5}$ (ii) $\frac{5}{4}$ (e) (i) $\frac{6}{7}$ (ii) $-\frac{7}{6}$ **4.** (a) $-\frac{5}{2}$ (b) $\frac{2}{5}$
5. (a) -1 (b) $y = -x + 9$ **6.** $\frac{1}{3}$, $y = \frac{1}{3}x + \frac{8}{3}$

EXERCISE 2·5, Pages 33-34

1. (a) $C \propto d$ (b) $C = kd$ **2.** (a) $l \propto w$ (b) $l = kw$ **3.** (a) $A \propto d^2$ (b) $A = kd^2$
4. (a) $V \propto d^3$ (b) $V = kd^3$ **5.** (a) $t \propto \sqrt{l}$ (b) $t = k\sqrt{l}$ **6.** (a) $l \propto t$ (b) $l = kt$
7. (a) $W \propto g$ (b) $W = kg$ **8.** (a) $L \propto v^2$ (b) $L = kv^2$ **9.** (a) $W \propto l^2$ (b) $W = kl^2$
10. 29.5 ft **11.** 58.7 ft **12.** 312 g **13.** 45 pounds **14.** (a) 11,900 cal.
(b) 45.4 ohms **15.** 4.5 amperes. **17.** 216:1331

EXERCISE 2·6, Page 36

1. $C = A + kP$ **2.** $C = S + kH$ **3.** $C = P + kn$ **4.** $C = L + km$ **5.** $V = P + kn$
6. $A = C + it$, \$224 **8.** \$772.50 **10.** (a) 896 ft/sec (b) −304 ft/sec **11.** (b) 9 sec

REVIEW EXERCISE 2, Pages 37-38

1. $y = \frac{3}{5}x - \frac{4}{5}$ **3.** 0, undefined, $\frac{4}{3}$, $-\frac{4}{3}$ **4.** $y = -\frac{5}{2}x + \frac{17}{2}$ **6.** (a) $C \propto n$ (b) $C = kn$
7. 16:1 **8.** \$300 **9.** 1:2 **10.** $2:\sqrt{5}$

CHAPTER 3

EXERCISE 3·1, Page 41

1. (b) **2.** (a) 120° (b) 210° (c) 270° (d) 240°
4. (a) The measure in degrees of angle XOY is 227.
(b) The measure in degrees of angle MON is 346. **5.** (a) $m°\angle AOF = 250$ (b) $m°\angle ANT = 310$

EXERCISE 3·2, Page 44

1. (a) 13 yd, 12.9 yd, 21.6 yd, 21.2 yd (b) 0.866, 0.643, 0.866, 0.707
(c) 7.5 yd, 15.3 yd, 12.5 yd, 21.2 yd (d) 0.5, 0.766, 0.5, 0.707
2. (a) (i) 33 yd (ii) 19 yd (b) (i) 40 yd (ii) 147.5 yd (c) (i) 94 yd (ii) 54 yd
(d) (i) 52 yd (ii) 52 yd **3.** (a) 45.3 yd (b) 0.906
(c) 21.2 yd (d) 0.423 (e) 61.5 yd (f) 57.4 yd (g) 9.3 yd (h) 90 yd.

EXERCISE 3·3, Pages 45-46

1. 45 miles west **2.** 500 feet south **3.** 153 miles N, −129 miles E **4.** 46 yd, no
5. 189 feet **7.** (a) 38, 22 (b) 23.8, 28.4 (c) 43, 74.5 **8.** (a) 75 mi N, 130 mi E
(b) 115 mi N, 96.5 mi E (c) 130 mi N, 75 mi E

EXERCISE 3·4, Page 48

1. (a) $0.8, 0.6, 1.\dot{3}$ (b) $0.8, -0.6, -1.\dot{3}$ (c) $-0.6, -0.8, 0.75$ (d) $-0.8, 0.6, -1.\dot{3}$
(e) $0.6, -0.8, -0.75$ (f) $-0.6, 0.8, -0.75$ **2.** $-0.87, 0.50, -1.75$ **3.** $-0.92, -0.44$
4. 13, 0.92, 0.38, 2.4; 13, 0.92, −0.38, −2.4; 9.2, −0.65, −0.76, 0.86

EXERCISE 3·5, Page 50

1. (a) $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$ (b) $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$ (c) $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$ (d) $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$
(e) $\frac{1}{2}, \frac{1}{2}\sqrt{3}, \frac{1}{3}\sqrt{3}$ (f) $\frac{3}{10}, \frac{1}{10}\sqrt{91}, \frac{3}{91}\sqrt{91}$ (g) $\frac{6}{85}\sqrt{85}, \frac{7}{85}\sqrt{85}, \frac{6}{7}$
2. (a) $\sqrt{41}$ (i) $\frac{4}{41}\sqrt{41}$ (ii) $\frac{5}{41}\sqrt{41}$ (iii) $\frac{4}{5}$ (b) $\sqrt{85}$ (i) $\frac{7}{85}\sqrt{85}$ (ii) $\frac{6}{85}\sqrt{85}$
(iii) $\frac{7}{6}$ (c) $\sqrt{39}$ (i) $\frac{5}{8}$ (ii) $\frac{1}{8}\sqrt{39}$ (iii) $\frac{5}{39}\sqrt{39}$ (d) $\sqrt{33}$ (i) $\frac{4}{7}$ (ii) $\frac{1}{7}\sqrt{33}$
(iii) $\frac{4}{33}\sqrt{33}$ (e) 12 (i) $\frac{12}{13}$ (ii) $\frac{5}{13}$ (iii) $\frac{12}{5}$ (f) $2\sqrt{2}$ (i) $\frac{1}{2}\sqrt{2}$ (ii) $\frac{1}{2}\sqrt{2}$
(iii) 1 (g) $3\sqrt{2}$ (i) $\frac{1}{3}$ (ii) $\frac{2}{3}\sqrt{2}$ (iii) $\frac{1}{4}\sqrt{2}$ (h) $2\sqrt{5}$ (i) $\frac{1}{3}\sqrt{5}$ (ii) $\frac{2}{3}$
(iii) $\frac{1}{2}\sqrt{5}$ **3.** $\frac{1}{7}\sqrt{33}, \frac{4}{33}\sqrt{33}$ **4.** $\frac{10}{149}\sqrt{149}, \frac{7}{149}\sqrt{149}$ **5.** $\frac{1}{3}\sqrt{5}, \frac{1}{2}\sqrt{5}$ **6.** $\frac{1}{11}\sqrt{21}, \frac{10}{21}\sqrt{21}$
7. $\frac{1}{12}\sqrt{95}, \frac{1}{7}\sqrt{95}$ **8.** $\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}$ **9.** infinitely many **10.** tangent **11.** sine

EXERCISE 3·8, Pages 54-55

1. .8290 **2.** 1.9626 **3.** .3746 **4.** 57.2900 **5.** .9336 **6.** .9994 **7.** .7431
8. .5299 **9.** 1.6003 **10.** .7071 **11.** .7071 **12.** 1.0000 **13.** .5878 **14.** .8090 **15.** .7265
16. $r = 81.4$, $s = 68.1$; $a = 40.2$, $b = 36.3$; $b = 18.1$,
$c = 40.0$; $s = 38.8$, $r = 37.5$ **17.** (a) 40°, 50°; 35°, 55°; 33°, 57°; 38°, 52° (b) 60.6, 45.3, 40.2, 43.7
18. 58°, 101, 119 **19.** 68°, 22°, 58.1

EXERCISE 3·9, Pages 56-59

1. 93.5 ft **2.** 12,380 ft **3.** 32.4 ft **4.** 137 ft **5.** 89.5″ **6.** 55° **7.** (a) 3600 ft
(b) 8.2 mph **8.** 8050 ft **9.** $h = 25.6$, $A = 870$ **10.** (a) $h = r \sin \theta$ (b) $\frac{1}{2} sr \sin \theta$ (c) 200
11. 190 **12.** (a) 225 (b) 1400 (c) 298 (d) not possible (e) 392 (f) 966
13. 745 **14.** 1440 **15.** $h = a \sin \theta$, $A = ab \sin \theta$, 884 **16.** 10.3 ft
17. 471 ft **18.** 23.3 ft **19.** 15.3, 15.3, 24.8

REVIEW EXERCISE 3, Page 59

2. 0.8 **3.** 47.8 ft **4.** 53° **5.** 197 yd

CHAPTER 4

EXERCISE 4·1, Page 62

1. $146.16 **2.** $53.90 **3.** (a) $35.62 (b) $58.28 (c) $154.98 **4.** (a) $37.60
(b) $518 (c) $2340 **5.** The first boy.

EXERCISE 4·2, Page 63

1. \$3700 **2.** 467 **3.** (a) \$9125 (b) \$2920 (c) 6484 (d) 110

EXERCISE 4·3, Pages 63-64

1. \$12,560 **2.** \$16,325 **3.** (a) \$20.50 (b) \$43.60 (c) \$4500

EXERCISE 4·4, Pages 64-65

1. A: \$457.69; B: \$306.88; C: \$268.04; D: \$652.60 **2.** \$94.12 **3.** \$1440 **4.** \$800
5. (b) *I* (i) \$15 (ii) \$60 (iii) \$105 (iv) \$135 *II* (i) \$10 (ii) \$40 (iii) \$100 (iv) \$140
III (i) \$35 (ii) \$50 (iii) \$65 (iv) \$75

EXERCISE 4·5, Page 66

1. $6\frac{3}{4}\%$ **2.** 5.8% **3.** (a) 11% (b) 6% (c) 3.56%

EXERCISE 4·6, Pages 67-68

4. (a) \$36 (e) $4\frac{1}{2}\%$ **5.** 15%

EXERCISE 4·7, Page 69

1. (a) \$61,000 (b) \$1372.50 (c) \$62,372.50 **2.** (a) \$925.31 (b) \$81,324.69
3. \$199,225 **4.** \$51,096.95 **5.** \$268.13, \$2825.87; 15%, \$4606.30;
\$946.31, \$17,066.31; \$4094.12, \$614.12; \$2608.70, \$391.30

REVIEW EXERCISE 4, Page 70

1. \$106 **2.** \$4000 **4.** \$30,805.60 **5.** \$7200, \$6336

CHAPTER 5

EXERCISE 5·1, Pages 73-74

1. 3, 3.1605 **2.** (a) 3.1429, 3.1409 (b) 3.1419 (c) 3.1419 (e) 0.0003 (f) 0.01%
(g) 3.1623 (h) 0.0207 (i) 0.7% **3.** (c) $\frac{128}{45} \approx 2.8444$ (d) $\frac{256}{75} \approx 3.4133$, (e) 2.9257
4. (a) 2.6667 (b) 3.4667 (c) 2.8952 (d) 3.3397 (e) 2.9760 (i) π (ii) 0.1656
5. (a) 2 (b) 2.6667 (c) 3.6667 (d) 3.4667 (e) 2.8 (f) 3.3952
6. (a) 2.5 (b) 2.8182 (c) 2.8285 **7.** (a) 0.432 (b) 0.43 **8.** (a) 0.434 (b) 0.43
9. (a) 0.435 (b) 0.43 **10.** (a) 0.435 (b) 0.44 **11.** (a) 0.444 (b) 0.44
12. (a) 0.445 (b) 0.44 **13.** (a) 0.445 (b) 0.44 **14.** (a) 0.976 (b) 0.98

EXERCISE 5·2, Pages 76-77

1. $w = \frac{V}{lh}$ **2.** $r = \sqrt{\frac{A}{\pi}}$ **3.** $C = \frac{5}{9}(F - 32)$ **4.** $h = \frac{2A}{B + b}, B = \frac{2A}{h} - b$ **5.** $D = 2t + d$

6. $R = \frac{E}{I}$ **7.** $t = \frac{A - P}{Pr}$ **8.** $p = \frac{A}{1 + rt}, r = \frac{A - P}{Pt}$ **9.** $h = \frac{A - 2\pi r^2}{2\pi r}$ **10.** $V = \frac{rT}{P}, r = \frac{PV}{T}$

11. $F = \frac{9}{5}C + 32$ **12.** $a = \frac{F}{m}$ **13.** $a = \frac{2S}{n} - l, n = \frac{2S}{a + l}$ **14.** $d = \frac{C - 4n}{20}$ **15.** $r = \frac{L}{2\pi h}$

16. $V = \sqrt{\frac{2gK}{w}}, W = \frac{2gK}{V^2}, g = \frac{WV^2}{2K}$ **17.** $a = \frac{S + \frac{1}{2}gt^2}{t}, g = \frac{2(at - S)}{t^2}$ **18.** $E = T(R + r), R = \frac{E}{T} - r$

19. (a) $S = 150$ (b) $C = 300$ **20.** 12 **21.** (a) 0.53 (b) 0.52
22. (a) $\frac{12}{7}$ (b) $\frac{40}{3}$ (c) -24 **23.** (a) 8900 mph (b) 34.5 hr **24.** (a) 450 ft (b) 4.47 sec
(c) $13\frac{1}{3}$ ft/sec/sec **25.** 6.26 sec **26.** 1730 ft **27.** 4.73 mi/sec

EXERCISE 5·3, Pages 79-80

1. unlike terms, monomials **2.** binomial expressions, polynomials **3.** unlike literal terms, monomials
4. trinomial expressions, polynomials **5.** like terms, monomials **6.** polynomial expressions

7. $7 \times (5 + 8 \div 2^2)$ **12.** 10 **13.** 1 **14.** $28\frac{1}{2}$ **15.** $39\frac{1}{3}$ **16.** $11\frac{1}{4}$ **17.** $13y - 36$
18. -24 **19.** $10y - 27\frac{1}{2}$ **20.** $\frac{7}{2}y + \frac{5}{2}\pi$ **21.** $\frac{8}{27}x - \frac{644}{243}$

EXERCISE 5·5, Pages 83-84

2. (a) 2 (b) $3r$ (c) $2b + c$ (d) $2m + x$ **3.** (a) $ha + t$ (b) $12yt + 8ys$ (c) $32y - 48x$
(d) $\frac{2}{3}y^2 - 9yz$ (e) $-18m^4 + 108m^2n^3$ (f) $-s^3 + s^4$ (g) $\frac{5}{2}w^3 - 20w^2 - 5w$ (h) $\frac{3}{2}z^3 + \frac{1}{4}z^2 - z$
(i) $b^5 + b^4 + 3b^3c - 2b^3d$ (j) $\frac{20}{21} - \frac{28r^2}{3s^3} + \frac{64s}{35r}$ **4.** (a) $a^2 - 4b^2$ (b) $3mx + 6nx + ym + 2yn$
(c) $2d^2 + 5dr + 3r^2$ (d) $25w^2 - 20wy + 4y^2$ (e) $196a^2 - 84ad + 9d^2$ (f) $w^4 - 2vw^2 + v^2$
(g) $a^2 - 4b^2$ (h) $9a^2 - 25b^2$ (i) $4d^2 - 16g^2$ (j) $y^2 - 2y^3 + y^4$ (k) $a^2 + 2ab + b^2$
(l) $m^2 + 4mn + 4n^2$ **5.** (a) $a(b + c + d)$ (b) $2x(y + z - 2w)$ (c) $\frac{3}{2}(x + b)$ (d) $\frac{4}{5}y(1 - 2y)$
(e) $\frac{2}{3}a(2a - b^2)$ (f) $\frac{5}{9}y(y^2 - y - 1)$ (g) $3x^2y(y + x - 1)$ (h) $8k^2(k^2 + 2kt + 3)$ **6.** doubled
7. quadrupled **8.** dm, $(d + 4)(m + 3)$ **9.** $(xy + 40x)$ miles **10.** $\frac{1}{2}(a - 2b)(3a + 2b)$

EXERCISE 5·6, Page 85

1. $x^2 + y^2 + 2xy$ **2.** $x^2 + 4y^2 + 4xy$ **3.** $4x^2 + 9y^2 + 12xy$ **4.** $9x^2 + 4y^2 + 12xy$
5. $64x^2 + 16y^2 + 64xy$ **6.** $100x^2 + 144y^2 + 240xy$ **7.** $\frac{4}{9}x^2 + \frac{1}{16}y^2 + \frac{1}{3}xy$ **8.** $\frac{9}{16}x^2 + \frac{9}{25}y^2 + \frac{9}{10}xy$
9. $x^2 + y^2 - 2xy$ **10.** $x^2 + 4y^2 - 4xy$ **11.** $9x^2 + 16y^2 - 24xy$ **12.** $16x^2 + 81y^2 - 72xy$
13. $25x^2 + \frac{1}{16}y^2 - \frac{5}{2}xy$ **14.** $36x^2 + \frac{4}{9}y^2 - 8xy$ **15.** $\frac{4}{9}y^2 + \frac{1}{25}z^2 - \frac{4}{15}yz$ **16.** $\frac{9}{16}x^2 + \frac{1}{4}y^2 - \frac{3}{4}xy$
17. $\frac{1}{9}x^2 + \frac{1}{16}y^2 + \frac{1}{6}xy$ **18.** $\frac{9a^2b^2}{c^2} + \frac{16x^2y^2}{z^2} + \frac{24abxy}{cz}$ **19.** $\frac{4}{9}m^2n^2 + \frac{9}{16}g^2h^2 - mngh$
20. $16x^2 + 25y^2 + 40xy$ **21.** $\frac{4}{9}x^2 + \frac{16}{25}y^2 - \frac{16}{15}xy$ **22.** $256x + 96y$ **23.** $144\pi x^2 + 9\pi y^2 - 72\pi xy$
24. 9801 **25.** 10,201 **26.** (a) 2401 (b) 2601 (c) 998,001 (d) 1,002,001
(e) 251,001 (f) 249,001 (g) 159,201 (h) 39,601

EXERCISE 5·7, Page 86

1. $x^2 - y^2$ **2.** $4x^2 - y^2$ **3.** $9x^2 - y^2$ **4.** $25x^2 - 16y^2$ **5.** $64x^2 - 81y^2$ **6.** $a^2x^2 - b^2y^2$
7. $\frac{4}{9}x^2 - \frac{25}{16}y^2$ **8.** $\frac{1}{4}\pi^2x^2 - \frac{9}{4}y^2$ **9.** $\frac{49}{81}x^2 - \frac{4}{25}y^2$ **10.** $\frac{1}{9}x^2 - \frac{1}{25}y^2$ **11.** $a^2b^2c^2 - m^2n^2$
12. $9r^2s^2t^2 - 16k^2l^2m^2$ **13.** $4u^2t^2 - 16r^2t^2$ **14.** $100v^2w^2 - 169u^2t^4$
15. $x^4y^6 - 9z^4$ **16.** $441k^2t^2 - 36r^2g^2$

EXERCISE 5·9, Page 88

2. $11m(3m - 1)$ **3.** $7ab^2(6a - 1)$ **4.** $r(12s - 7t)$ **5.** $b^2(a - 3b)$ **6.** $6yz(4xy^2 - xz + 8y)$
7. $3w(5xw - 3)$ **8.** $(n - w)(5m - 3p)$ **9.** $3x(8z^2 - 5x + 6z)$ **10.** $(y - 2z)(3x + 5)$
11. $(z - w)(xy + 7l)$ **12.** $7x^2y(2y^2 - 4 + 7xy)$ **13.** $xw(z^2 - w)(1 + xw^2)$ **14.** $5xy(1 - 2xy + 4x^2y^2)$
15. $z(3a - 2b + c)(z - w^2 + w)$ **16.** $(x + y + z)(y + w)$ **18.** $y - z$
19. $1 - \frac{yc}{b}$ **20.** $1 - c$ **21.** $y - 1$ **22.** $1 + \frac{z}{y} + \frac{w}{y}$ **23.** $\frac{1}{y^2} - \frac{2}{y}$
24. $\frac{4a - 8b}{b - c}$ **25.** $-\frac{y}{x}$ **26.** 1 **27.** $\frac{(ma + th)(i + s)}{(f + un)(e + h)}$

EXERCISE 5·10, Pages 89-90

1. $(x + y)(x - y)$ **2.** $(m + n)(m - n)$ **3.** $(2b + 5d)(2b - 5d)$ **4.** $(3k + 4h)(3k - 4h)$ **5.** $(6z + 7y)(6z - 7y)$
6. $(10y + 12w)(10y - 12w)$ **7.** $(11r + 3v)(11r - 3v)$ **8.** $(15t + 6s)(15t - 6s)$ **9.** $(rt + vu)(rt - vu)$
10. $(8x + 12y)(8x - 12y)$ **11.** 240 **12.** 660 **13.** -480 **14.** 220 **15.** 10,740
16. 93,000 **17.** 352 **18.** -320 **20.** (a) 93π (b) 35π (c) 108π

21. (a) 234 ft (b) 32 ft|sec|sec (c) 1220 **22.** (a) $0.24R(C + c)(C - c)$ (b) 4860 cal (c) 30,643.20 cal
23. (a) $(x^2 + 1)(x + 1)(x - 1)$ (b) $(x^2 + y^2)(x + y)(x - y)$ (c) $(9x^2 + y^2)(3x + y)(3x - y)$
(d) $(100m^2 + 1)(10m + 1)(10m - 1)$ **24.** (a) $(a - b + y)(a - b - y)$ (b) $(a + b + y)(a + b - y)$
(c) $(2r - s + t)(2r - s - t)$ (d) $(m - n + h)(m - n - h)$ (e) $(3g + 4h + t)(3g + 4h - t)$
(f) $(x + y - z + w)(x + y + z - w)$ **25.** (a) $a + b$ (b) $a - 2b$ (c) $x + 2y$ (d) $x + 3y$
26. (a) $2(3t + 5r)(3t - 5r)$ (b) $8(2m + r)(2m - r)$ (c) $27(m + 2n)(m - 2n)$ (d) $-5mn^2$
(e) $5(2r + 5s)(2r - 5s)$ (f) $m^3n^3(n + m)(n - m)$

EXERCISE 5·11, Page 91

7. $(a - b)(c + 4)$, $(m + 3)(m + n)$, $(a + 5)(a + b)$, $(p + 1)(p + m)$, $(r + 1)(r + s)$
8. $(a + b)(m + 3)$ **9.** not possible **10.** $(m + n)(5a + 4)$ **11.** $(m + n)(5a + 4)$ **12.** $(r - s)(st + 4)$
13. $(ma + th)(is + fun)$ **14.** $(4r + 3p)(t + 20p)$ **15.** $(4r + 3p)(t + 20p)$ **16.** $(8am + 9b)(2a + 3)$
17. $(8am + 9b)(2a + 3)$ **18.** 2, 2 **19.** (a) $y - w$ (b) $m + s$

EXERCISE 5·12, Page 95

1. 21 **2.** 33 **3.** 21 **4.** 11 **5.** 121 **6.** 231 **7.** $6x^2 + 3$ **8.** $4ab - 2b^2$ **9.** $14a + 4$
10. $14 - x$ **11.** $8y - 9$ **12.** $5 - 7c^2$ **13.** $9b^2 - 27b + 2$ **14.** $30a^2b^2 + 5ab - 4$ **15.** $4x + 2$
16. $3x + 2$ **17.** $-7x + 6$ **18.** $62x - 6$ **19.** $x^2 + 4x + 8$ **20.** $2x^2 + 7x + 4$
21. $(a + 3)$ with remainder -2 **22.** $x^2 + 2x + 4$ **23.** $m + 3$ **24.** $(3n + 4)$ with remainder 11
25. $x^2 + 3x + 9$ **26.** $(-\frac{1}{2}x^2 + x - 1)$ with remainder 1 **27.** $(2x^3 - \frac{4}{3}x^2 + \frac{2}{9}x - \frac{13}{27})$ with remainder $\frac{53}{27}$
28. $(2x^2 - \frac{1}{2})$ with remainder 2 **30.** $x + 9$ **31.** $x + 9$ **32.** $x + 7$ **33.** $x - 7$
34. $x + 6$ **35.** $x + 6$ **36.** $x + 4$ **37.** $x + 4$ **38.** $x + 5$ **39.** $x - 5$ **40.** $(x - 8)(x + 8)$

EXERCISE 5·13, Page 96

1. 252 **4.** $m - 1$ **10.** $7s + 2$ **12.** 3 **13.** $\frac{4}{3}$

EXERCISE 5·14, Page 97

1. $-\frac{19}{7}$ **2.** -2 **3.** $-\frac{3}{2}$ **4.** $-\frac{17}{8}$ **5.** $\frac{55}{9}$ **6.** π **7.** 2 **8.** $10'' \times 18''$, $9'' \times 20''$
9. base = height = 17", top = 15", area = 272 sq in **10.** 4 in, 6 in., 24 sq. in.

EXERCISE 5·15, Page 98

1. (a) 41 (b) $\frac{3}{4}$ (c) $4\frac{1}{5}$ (d) 0 (e) 49 (f) $31\frac{1}{3}$ (g) $5\frac{1}{4}$ (h) 23 **2.** (a) 17 (b) 6
(c) 16 (d) 64 (e) 16 (f) 26 **3.** (a) 17 (b) 6 (c) 16 (d) 64 (e) 16 (f) 26
4. (a) 3 (b) 7 (c) 0 (d) 2 (e) 36 (f) $7\frac{1}{2}$ (g) $9\frac{1}{4}$ (h) $11\frac{7}{8}$

EXERCISE 5·16, Page 100

1. $>$ **2.** $<$ **3.** $<$ **4.** $<$ **5.** $=$ **6.** $>$ **7.** $>$ **8.** $<$ **9.** $<$ **10.** $<$ **11.** $<$ **12.** $<$
15. (a) $>$ (b) $<$ (c) $<$ (d) $>$ (e) $<$ (f) $>$ **16.** (a) $<$ (b) $<$ (c) $<$ (d) $>$
(e) $<$ (f) $>$ **17.** (a) $>$ (b) $>$ (c) $>$ (d) $<$ (e) $>$ (f) $<$

EXERCISE 5·17, Page 102

1. $x \leq 7$ **2.** $x < 2$ **3.** $x > -\frac{7}{2}$ **4.** $x < -5$ **5.** $x > \frac{16}{3}$ **6.** $x < -\frac{17}{11}$
7. $x \leq \frac{9}{8}$ **8.** $x \leq \frac{1}{34}$ **9.** $x \geq \frac{93}{46}$ **10.** All $x \in R$ **11.** $x < -\frac{5}{3}$
12. $x > \frac{21}{8}$ **13.** $x < -\frac{2}{11}$ **14.** $x \geq -39$ **15.** $x \geq 5$ **16.** ϕ **17.** ϕ

REVIEW EXERCISE 5, Pages 103-104

1. $-\frac{2}{21}$ **2.** 24 **3.** $\frac{14}{15}$ **4.** $-\frac{39}{40}$ **5.** $-\frac{1}{3}$ **6.** $\frac{4}{3}$ **7.** 4.14
8. 0.57 **9.** 0.48 **10.** 41.13 **11.** $\frac{25}{3}$ **12.** $\frac{4}{3}$ **13.** $9\frac{1}{2}$ in, 27 in, $20\frac{1}{2}$ in
14. (a) $4\frac{4}{5}$ ohms (b) 6 ohms **15.** (a) $\frac{1}{8}$ (b) $22.2d$ **16.** (a) $18a^2 + 30ad + 8d^2$
(b) $4b^2 - 9d^2$ (c) $m^2 - 4n^2 + 4n^2$ **17.** (a) $a(x + y + z)$ (b) $\frac{3}{5}t(t + 1)$ (c) $2(y - 3z)(x + 3)$
(d) $(2a + 3b)(2a - 3b)$ **18.** (a) $2a - 4$ (b) $2a + 1$ **19.** (a) yes (b) no **20.** 5 **21.** $\frac{7}{29}$ **22.** $x < \frac{1}{3}$

CHAPTER 6

EXERCISE 6·1, Pages 108-109

1. one **2.** no, no, **3.** yes, no **4.** $y = -\frac{2}{3}x + 12$, $y = -\frac{2}{3}x + 12$, yes, yes, $2x + 3y = 24$
5. (a) unequal, intersect (b) equal, unequal, are parallel (c) equal, equal, are coincident, multiple
6. The equations must be consistent. See 5.(a) **7.** consistent **8.** consistent
9. inconsistent **10.** dependent **11.** 2, 2 **12.** −13, 19 **13.** −3, −3 **14.** 0, 4 **15.** (3, −6)
16. (3, 4) **17.** (1.6, 1.3), **18.** (−0.7, 2.5) **19.** (13, 31) **20.** (−2.5, −4)

EXERCISE 6·2, Page 111

3. (a) open (b) $y < \frac{2}{3}x + 2$ (c) $y < \frac{2}{3}x + 2$ **5.** (c), (d), (g), (h)
6. open: (a), (b), (e), (f); closed: (c), (d), (g), (h)

EXERCISE 6·3, Pages 113-114

1. $y < \frac{2}{3}x + 2$, $y \geq 4 - 3x$, $y < -2$, $x \leq 4$, $y < -x - 2$, $x > -2$
5. (a) T (b) T (c) F (d) F (e) F (f) F (g) F (h) F

EXERCISE 6·4, Page 116

1. (8, 7) **2.** (4, −2) **3.** (6, −3) **4.** (−5, −2) **5.** (−1, −2) **6.** (−2, −2)
7. (0.5, 0.25) **8.** (−3.9, 2.5) **9.** (−1.3, 0.4) **10.** 50, $50

EXERCISE 6·5, Page 117

1. (1, −2) **2.** (12, 3) **3.** $(\frac{46}{15}, \frac{1}{15})$ **4.** (16, −9) **5.** (51, 49) **6.** (19, −7) **7.** $(\frac{5}{8}, 1\frac{3}{8})$
8. (0.3, 0.7) **9.** $(\frac{12}{5}, \frac{18}{5})$ **10.** (8, 19) **11.** (−7, 2)

EXERCISE 6·6, Page 119

1. $3x + 9y = 27$ **2.** $-10x + 5y = 30$ **3.** $-24x - 4y = -36$ **4.** $13y = 65x + 52$ **5.** $24x = 6y - 42$
6. $-3x + y = -4$ **7.** (−3, −1) **8.** (−13, 15) **9.** (12, −7) **10.** $(3, \frac{8}{3})$ **11.** (3, −2)
12. (1, 2) **13.** (10, 3) **14.** (6, 8)

EXERCISE 6·7, Pages 119-120

1. $6x + 4y = 14$ **2.** $6y - 9x = 12$ **3.** $12x - 9y = 15$ **4.** $21x - 39y = -9$ **5.** $30y - 78x = 24$
6. $77x - 42y = -28$ **7.** (a) 20 (b) 6 (c) 6 (d) 12 (e) 24 (f) 66 (g) 21 (h) 15
(i) 24 **8.** (−3, 11) **9.** (4, −9) **10.** (11, 12) **11.** $(\frac{167}{2}, -\frac{119}{2})$ **12.** (3, −2) **13.** (1, 2)
14. (11.4, 8.2) **15.** $(-\frac{14}{39}, \frac{20}{13})$

EXERCISE 6·8, Pages 121-122

1. $2w + 2l = 88$, $l = 2w + 8$ **2.** $2l + b = 55$, $l = 2b$ **3.** $m = 2n + 2$, $2m + 3n = 221$
4. $x + y = 900$, $0.05x + 0.10y = 72.50$ **5.** $x + y = 90$, $x = 3y + 4$ **6.** $x + y = 58$, $x + 10 = 2(y + 10)$
7. $x + y = 3450$, $5x = 2y - 425$ **8.** (1) 12, 32 (2) 11, 22 (3) 64, 31 (4) $350, $550
(5) $21\frac{1}{2}°$, $68\frac{1}{2}°$ (6) 16, 42 (7) 925, 2525 **9.** $80, $120 **10.** 32, 68 **11.** $130, $170
12. $1\frac{1}{3}$ min, 8 mi/min

REVIEW EXERCISE 6, Page 122

1. $(\frac{3}{2}, \frac{1}{2})$ **2.** Golf balls are 50¢ each; tennis balls are 3 for $1.75 (59¢ each). **3.** $y = -3$, $x = 2$

CHAPTER 7

EXERCISE 7·2, Pages 124-125

3. $m(\overline{AB}) = 5$, $m(\overline{DE}) = 9$, $m(\overline{FG}) = x_2 - x_1$, $m(\overline{KL}) = q - r$ **4.** 7 **5.** 3 **6.** $|x_1 - x_2|$

7. $|x_1 - x_2|$ **8.** $|x_1 - x_2|$, parallel **9.** $m(\overline{CD}) = 8$, $m(\overline{EF}) = 7$, $m(\overline{AB}) = 6$, $m(\overline{KL}) = |y_1 - y_2|$
10. (a) 4 (b) 2 (c) 9 (d) $|y_1 - y_2|$ (e) $|p - n|$ (f) $|h - k|$

EXERCISE 7·3, Pages 126-127

1. $\sqrt{65}$ **2.** $m(\overline{MN}) = 5$, $m(\overline{NP}) = 5$, $m(\overline{MP}) = 5\sqrt{2}$ **3.** $m(\overline{EF}) = 9$, $m(\overline{FG}) = 7$, $m(\overline{EG}) = \sqrt{130}$
4. $m(\overline{HJ}) = 8$, $m(\overline{JK}) = 11$, $m(\overline{HK}) = \sqrt{185}$ **5.** $\sqrt{157}$ **6.** (a) $4\sqrt{10}$ (b) $\sqrt{181}$
(c) $\sqrt{89}$ (d) $\sqrt{289}$ **7.** (a) $R(x_1, y_2)$ (b) $|x_2 - x_1|$ (c) $|y_2 - y_1|$
(d) $\sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$ **8.** (a) $\sqrt{5}$ (b) 13 (c) $6\sqrt{2}$ (d) $6\sqrt{2}$
(e) $\sqrt{181}$ **9.** (a) $\sqrt{5}$ (b) 13 **11.** $\sqrt{26} + \sqrt{58} + \sqrt{148}$ **12.** 29.320, 26.912, 23.419

EXERCISE 7·4, Pages 128-129

1. $(1, 3), (2, -2\frac{1}{2}), (-4, -3); (2, 4), (\frac{1}{2}, -4\frac{1}{2}), (-4, 1\frac{1}{2})$
2. $(-\frac{1}{2}, \frac{3}{2}), (-1, -\frac{1}{2}), (3, 1.375), (2.87, -.31), (-\frac{1}{2}, -2\frac{1}{2}), \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$
3. (a) $(-1, 2\frac{1}{2})$ (b) (2.92, 0.93) (c) (−0.61, −2.38) (d) $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$
(e) $\left(\frac{a+e}{2}, \frac{b+d}{2}\right)$ (f) $\left(\frac{\sqrt{2}+\sqrt{6}}{2}, \frac{\sqrt{6}+\pi}{2}\right)$ **4.** (2, 1), (2, 1) **5.** $(2, 4), (-\frac{1}{2}, 2), (\frac{1}{2}, 0)$

EXERCISE 7·5, Page 130

1. no **2.** no **3.** no **4.** (a) yes (b) yes (c) no (d) no

EXERCISE 7·6, Page 131

1. 5, 13, $8\sqrt{5}$, no **2.** $4\sqrt{5}$, $2\sqrt{5}$, $6\sqrt{5}$, yes **3.** $\sqrt{37}$, $\sqrt{101}$, $2\sqrt{5}$, no **4.** $2\sqrt{2}$, 5, $\sqrt{61}$, no **5.** no
6. yes **7.** Exclude $(4, -\frac{7}{2})$. **8.** yes

REVIEW EXERCISE 7, Page 132

1. (a) $\sqrt{41}$ (b) $2\sqrt{13}$ (c) $9\sqrt{2}$ **2.** (a) $(\frac{1}{2}, 1)$ (b) (−0.7, 0.8) **3.** (a) yes (b) yes
5. $(-1, 2)$ Radius is $\sqrt{65}$. Hence 4 other points are $(-1 + \sqrt{65}, 2)$, $(-1 - \sqrt{65}, 2)$, $(-1, 2 + \sqrt{65})$, $(-1, 2 - \sqrt{65})$. **6.** No, they could be distinct parallel lines.

CHAPTER 8

EXERCISE 8·1, Page 136

5. $[-7, 8], [8, 7], [3, -8], [-4, -4]$ **7.** (a) $[(3, 2), (7, 5)]$ (b) $[(3, 2), (0, 7)]$ (c) $[(3, 2), (-1, -4)]$
8. $\overrightarrow{AB}, \overrightarrow{PM}, \overrightarrow{TR}$ **9.** (a) $[(3, 4), (-5, -1)], [(3, -4), (-5, 2)], [(2, 5), (-6, -3)]$
(b) $[-8, -5], [-8, 6], [-8, -8]$ **10.** (a) 13 (b) $2\sqrt{52}$ (c) $3\sqrt{10}$ (d) $2\sqrt{58}$ (e) $\sqrt{89}$

EXERCISE 8·2, Pages 139-140

10. $[2, -9]$ **11.** $[-10, -16]$ **12.** $[4, 10]$ **13.** $[0, 3]$ **14.** $[-12, 0]$ **15.** $[7, 9]$ **16.** $[-1, 6]$
17. $[1, -13]$ **18.** $[0, 0]$ **19.** $[4, 5]$ **20.** $[12, 15]$ **21.** $[10, 0] + [0, 8] = [10, 8]$
22. $[0, 7.5] + [7.5, 0] = [7.5, 7.5]$ **23.** $[3, 5] + [6, -5] = [9, 0]$

EXERCISE 8·4, Page 146

1. 23 lb 48° to 30 lb vector **2.** 64 mi 80° to 24 mi vector **3.** 133 mph 73° to 100 mph vector
4. 89 ft/sec/sec 85° to 89 ft/sec/sec vector **5.** 76 lb 50° to 33 lb vector **6.** 208 mi 45° to 120 mi vector
7. 69 ft/sec/sec 38° to 70 ft/sec/sec vector **8.** 42 lb 54° to 58 lb vector **9.** 5.26, 103° **10.** 36, 16°
11. 6.1 at 99° to $\overrightarrow{HG}$ *or* 16.8 at 31° to $\overrightarrow{HG}$ **12.** 29.2 mi 19° west of north **13.** 36 lb **14.** 14.7 mi, 50°

EXERCISE 8·5, Page 147

1. 57.4, 39.2 **2.** 29.7, 18.5 **3.** 15, 8, 28° **4.** 15.1, 23.5, 50° **5.** 77.7, 78°, 16.2 **6.** 889 mi

EXERCISE 8·6, Pages 149-150

1. 230 mph N17°W **2.** 68 mi N34°E **3.** 14.6 mi **4.** 996 mph S37°E **5.** 845 pounds **6.** 357 g

REVIEW EXERCISE 8, Page 150

1. (a) [−4, 11] (b) [−4, 19] **4.** 650 mph N47°E **5.** B **6.** 45 knots S28°W **7.** Speed of tide = 19.5 knots, speed of submarine = 26 knots

CHAPTER 9

EXERCISE 9·2, Page 154

1. (a) $66\frac{2}{3}$ (b) 55.3 (c) 146 (d) 150 (e) 200 **2.** (a) 20 (b) 24.5 (c) 97.5 (d) 15.36 (e) 1.95 (f) 105 (g) 165.30 (h) 99.75 **3.** (a) 70 (b) 40 (c) 650 (d) 200 (e) 169.8 (f) 118 **4.** (a) 25.16 (b) 27.1 (c) 16.32 (d) 190.9 (e) 172.7 (f) 68.8

EXERCISE 9·3, Page 155

1. loss of $212 **2.** loss of $338.75 **3.** $413.34 **4.** $22,000

EXERCISE 9·4, Page 158

1. (1) **2.** $1222 **3.** $1342.86 **4.** (a) $10, $26.20, $161.20, $110, $422 (b) 13.3%, 19.8%, 21.1%, 22.1%, 29.2% (c) 15.4%, 24.6%, 26.7%, 28.4%, 41.2% **5.** (a) $396.80 (b) $76.80 (c) 19.3% **6.** (a) $4.50 (b) $30 (c) 13% **7.** (a) $4.16 (b) $2.34 (c) 56% **8.** $.72, $2.88, $33\frac{1}{3}$%; $28.35, $12.15, 43%; $22.74, $59.14, 38%; $57.92, $14.48, 20%; $21.27, $27.27, 28%; $22.06, $30.56, 28%.

EXERCISE 9·5, Pages 160-161

1. $26,900, $3,700, $350 **9.** (a) 39.4%, 55.8% (b) 26.1%, 55.6%

EXERCISE 9·6, Page 163

1. $25.61 **2.** $104.04 **3.** $2.52 **4.** $33.01 **5.** $16.20 **6.** $40.89 **7.** $286.38 **8.** $197.34 **9.** $297.79 **10.** $785.09 **11.** $73.47 **14.** 36% **15.** 40% **16.** 30% **17.** $44\frac{4}{9}$%

EXERCISE 9·7, Pages 164-165

1. 41.5% **2.** 40% **3.** 75% **4.** 49.6% **5.** 50.5% **6.** 19% **7.** $370 **8.** $893 **9.** $1110 **10.** $1370 **11.** A. 56.8% B. 56.8% C. 56.8% **14.** 1965 Dodge: $1057.50, $951.75, $856.58; 1968 Dodge: $2475, $2227.50, $2004.75: 1967 Plymouth: $1989, $1790.10, $1611.09 **15.** 63 Chev. Sedan

EXERCISE 9·8, Page 168

1. $7.60, $372.40 **2.** $452 **3.** $.71, $34.79 **4.** $9.01, $441.59 **5.** $762.40 **6.** $3.91, $394.41 **7.** $44.88, $852.77 **8.** $261.00

EXERCISE 9·9, Pages 172-173

1. (a) J. R. T. Togans Ltd., R. T. Morris (b) $300 (c) Nov. 20, 1969 (d) Sept. 23, 1970 (e) $9\frac{1}{2}$% (f) 10 months **2.** (a) Feb. 20, 1970 (b) May 23, 1972 (c) March 20, 1973 (d) May 18, 1970 (e) Feb. 28, 1969 (f) May 13, 1969 (g) March 3, 1971

4. (a) $331.70 (b) $763.13 (c) $1230.90 (d) $1047.67 (e) $665.64 **5.** $676.85
6. $504.97 **7.** $323.97 **8.** (a) $325.30 (b) $753.45 (c) $1036.73 (d) $980.65
(e) $642.59 **9.** (a) $300.36 (b) $861.11 (c) $427.15
(d) $1700 (Aug. 30 is after maturity date so note is past due.)

REVIEW EXERCISE 9, Pages 173-174

1. $42,500 **2.** (a) $184.62 (b) $64.62 (c) 55% **3.** 31% **4.** (a) $31,500, $15,250, $2,750
5. B **6.** Feb. 28, 1972 **7.** $744.95 **8.** $955.66

CHAPTER 10

EXERCISE 10·1, Pages 178-179

6. 58 mills **7.** $8,411,680 **8.** $13,109,339 **9.** $85,842.56 **11.** $5,426,986.19

EXERCISE 10·2, Page 180

1. $606.94 **2.** $177.13 **3.** $821.15 **4.** $810.40 (based on tax bill for question 3) **5.** $583.80
6. $781.63 (a) $484.61 (b) $161.54

EXERCISE 10·3, Pages 181-182

3. (a) $94.25 (b) $6.50 (c) 6.9% over four years (or 1.725% per annum). (d) $3\frac{1}{2}$% per annum
4. (a) $216

EXERCISE 10·4, Page 183

3. $322 **4.** (a) $546.99 (b) 35% **5.** $1368 **6.** (b) $3473.72 (d) $3368.46

EXERCISE 10·5, Page 186

1. (a) 5 mills, 8.25% (b) $430 (c) $468.38 **4.** $549.86

REVIEW EXERCISE 10, Page 187

1. 63.5 mills **2.** $58.86 **3.** (a) $540.75, $630.75 (b) 30 cents per foot (c) $2\frac{1}{2}$% p.a.
(d) $5\frac{1}{2}$% **4.** (a) $1380 (b) $1260 **5.** $1\frac{2}{3}$ mills

CHAPTER 11

EXERCISE 11·1, Page 190

2. (c) $\sqrt{x} \notin R$ for $x < 0$ (d) $\sqrt{-4} \notin R$ (h) $\sqrt{-9} \notin R$ **3.** (a) 5 (b) 9 (c) 4 (d) 2
(e) $\frac{1}{3}$ (f) $-\frac{2}{3}$ (g) $\frac{9}{10}$ (h) $-\frac{1}{6}$ (i) $\frac{10}{3}$ (j) 1.2 (k) 1.2 (l) 1.5 (m) $5x$ (n) ab^2
(o) $5c$ (p) $7d$

EXERCISE 11·2, Pages 192-193

1. 30 **2.** 36 **3.** 84 **4.** $\frac{2}{3}$ **5.** $\frac{39}{8}$ **6.** $\frac{13}{7}$ **7.** $\frac{13}{24}$ **8.** $\frac{83}{36}$ **9.** $xy^2\sqrt{y}$ **10.** $4x\sqrt{2}$
11. $2ab\sqrt{21}$ **12.** $mn^2\sqrt{11}$ **13.** $mn^3\sqrt{58}$ **14.** $13\sqrt{77}$ **15.** $5my\sqrt{5my}$ **16.** $(a-b)\sqrt{b}$
17. $\frac{4}{7}\sqrt{3}$ **18.** $(x+2)y\sqrt{y}$ **19.** $(2a+b)\sqrt{3}$ **20.** $(x+7)\sqrt{b}$ **21.** not reducible **22.** $\pm\frac{3}{2}$
23. $\pm\frac{4}{3}\sqrt{3}$ **24.** $\pm\sqrt{13}$ **25.** ϕ **26.** $\pm\frac{9}{5}\sqrt{5}$ **27.** $\pm\frac{8}{13}$ **28.** ϕ **29.** ϕ **30.** $\pm\frac{5}{3}\sqrt{\frac{5}{3}}$
31. ± 2 **32.** $\sqrt{9a^2bc}$ **33.** $\sqrt{196b}$ **34.** $\sqrt{\frac{1}{9}mn}$ **35.** $\sqrt{225x^4yz}$ **36.** $\sqrt{81x^3y^3z^4}$
37. $\sqrt{242a^5b^7}$ **38.** $\sqrt{\frac{4a}{125}}$ **39.** $\sqrt{\frac{27}{512}ax^5}$

EXERCISE 11·3, Pages 195-196

1. 841 **2.** 1369 **3.** 44.89 **4.** 608,400 **5.** 79.21 **6.** 3.742 **7.** 6.633 **8.** 7.348
9. 9.327 **10.** 9.695 **11.** 23 **12.** 71 **13.** 89 **14.** 3.6 **15.** 7.2 **16.** $\frac{1}{2}\sqrt{10}$
17. $\frac{3}{2}\sqrt{2}$ **18.** $\frac{1}{2}\sqrt{62}$ **19.** $\frac{2}{3}\sqrt{15}$ **20.** $\frac{2}{5}\sqrt{10}$ **21.** $\sqrt{3}$ **22.** $\frac{1}{y}\sqrt{xy}$ **23.** $y\sqrt{3y}$
24. $\frac{1}{x}\sqrt{5x}$ **25.** $z^2\sqrt{xyz}$ **26.** $\frac{1}{b}\sqrt{5b}$ **27.** $\frac{1}{n}\sqrt{mn}$ **28.** $\frac{3\sqrt{a^2-b^2}}{2(a+b)}$ **29.** $\frac{\sqrt{2a^2-3ab-2b^2}}{a-2b}$
30. $\frac{\sqrt{6a^2+5ab-6b^2}}{2a-3b}$ **31.** 0.71 **32.** 1.34 **33.** 0.45 **34.** 0.77 **35.** 0.38 **36.** 0.18
37. 4.04 **38.** 2.54 **39.** 4.57 **40.** 4.24 **41.** 25.45 **42.** 92.94 **43.** $xy\sqrt{6}$ **44.** $6\,m^2n^2$
45. $mnab$ **46.** $xy\sqrt{wz}$ **47.** $12\sqrt{a^2-b^2}$ **48.** $a+b$ **49.** $12a-6b$ **50.** $xa^2\sqrt{x}$

EXERCISE 11·6, Page 202

1. Answers more than 4 questions correctly. **2.** Answers 3 questions correctly. **3.** No
5. $A = IV, B = II, C = I$ **6.** (a) 16 (b) 15 (c) \$4000 (d) \$100 (e) 4 (f) \$50,000
8. $\{(A, s) | A = s^2, A, s \in R^+\}$ **9.** $\{(V, s) | V = s^3, V, s \in R^+\}$

EXERCISE 11·7, Page 205

1. 40-foot position **2.** 0.7g, 0.4g, 0.6g, 0.3g **4.** (d) 10 **5.** (b) $189\frac{17}{27}$, $142\frac{2}{9}$, $106\frac{2}{3}$
(e) $\{(x,y) | y = 80(\frac{3}{4})^x, x, y \in R\}$

EXERCISE 11·8, Page 207

1. (a) $10^{0.1}$ (b) $10^{0.65}$ (c) $10^{0.9}$ (d) $10^{0.93}$ **2.** (a) 2.2 (b) 1.8 (c) 5.6 (d) 4.4
3. (a) 9.7 (b) 9.0 (c) 6.1 (d) 9.6 **4.** (a) 3.0 (b) 3.4 (c) 1.2 (d) 3.5
5. (a) 6.6 (b) 1.7 (c) 5.0 (d) 4.0 **6.** (a) 1.8 (b) 5.6 (c) 1.9 (d) 130
(e) 0.5 (f) 0.5 (g) 1.25 (h) 13 (i) 5.0 (j) 1.3

EXERCISE 11·9, Page 209

1. (a) 3.56×10^2 (b) 4.7645×10^4 (c) 7.641×10 (d) 3.65×10^{-1} (e) 5.67×10^{-2} (f) 8.63
(g) 1.403×10 (h) 7×10^{-5} (i) 8.765431×10^5 (j) 4.611×10 (k) 3.2×10^{-2} (l) 7.6×10^{-3}
2. (a) 341 (b) 0.0862 (c) 186,000 (d) 88 (e) 96,000,000 (f) 0.0000041 (g) 7632
(h) 0.034567 (i) 7610 (j) 33,300 (k) 0.00012 (l) 9.6
3. (a) $10^{1.6}$ (b) $10^{0.4}$ (c) $10^{2.67}$ (d) $10^{1.77}$ (e) $10^{3.46}$ (f) $10^{5.55}$ (g) $10^{0.09}$
(h) $10^{1.22}$ **4.** (a) $10^{.8} \times 10^{-2}$ (b) $10^{.8} \times 10^{-5}$ (c) $10^{.4} \times 10^{-3}$ (d) $10^{.6} \times 10^{-1}$
(e) $10^{.8} \times 10^{-4}$ (f) $10^{.4} \times 10^{-1}$ (g) $10^{.8} \times 10^{-3}$ (h) $10^{.3} \times 10^{-2}$ (i) $10^{.68} \times 10^{-3}$
5. (a) $10^{2.55}$ (b) $10^{4.67}$ (c) $10^{1.88}$ (d) $10^{-.44}$ (e) $10^{-1.24}$ (f) $10^{.93}$ (g) $10^{1.15}$
(h) $10^{-4.15}$ (i) $10^{5.94}$ (j) $10^{1.67}$ (k) $10^{-1.5}$ (l) $10^{-2.12}$
6. (a) 1.6×10^4 (b) 2.3×10^3 (c) 1.0 (d) 3.1×10^{-2} (e) 1.2 (f) 85 (g) 20
(h) 1.5×10^2 (i) 0.46 (j) 4.2×10^{-2} (k) 1.7×10^{-3} (l) 5.0×10^{-1} (m) 8.5 (n) 42

EXERCISE 11·10, Page 211

1. (a) $10^{.5775}$ (b) $10^{.6702}$ (c) $10^{1.7292}$ (d) $10^{3.8854}$ (e) $10^{-2.1163}$ (f) $10^{-.3820}$ (g) $10^{-1.1186}$
(h) $10^{-3.5003}$ (i) $10^{-2.9547}$ **2.** (a) 43.5 (b) 568 (c) 6970 (d) 10.168 (e) 0.0255
(f) 0.00316 (g) 0.263 (h) 0.000514 (i) 0.000631 (j) 5,450,000 **3.** (a) 1790
(b) 131,000 (c) 31,800 (d) 30.6 (e) 2.66 (f) 4.65 (g) 0.0042 (h) 3.32
(i) 1740 (j) 18.1 (k) 87.5 (l) 1680

EXERCISE 11·11, Pages 212-213

1. (a) $\log_{10}81 = 1.9085$ (b) $\log_{10}628 = 2.7980$ (c) $\log_{10}.0455 = -2 + .6580$
(d) $\log_{10}.0455 = -1.342$ (e) $\log_{10}.00345 = -3 + .5378$ (f) $\log_{10}478 = 2 + .6794$
3. (a) $\log_{10}394 = 2.5955$ (b) $\log_{10}28300 = 4.4518$ (c) $\log_{10}1.45 = 0.1614$

(d) $\log_{10}0.00479 = -2.3197$ (e) $\log_{10}0.0697 = -1.1568$ (f) $\log_{10}4.17 = 0.6201$
4. (a) $10^{1.9445} = 88$ (b) $10^{1.6513} = 44.8$ (c) $10^{2.7782} = 600$ (d) $10^{-2+.6021} = .04$
(e) $10^{-1.301} = .05$ (f) $10^{2.8235} = 666$ **5.** (a) 3930 (b) 50.6 (c) 5.88 (d) 0.750
(e) 0.251 (f) 0.0794

REVIEW EXERCISE 11, Pages 213-214

1. (a) $\sqrt{30}$ (b) 24 (c) $3nm^3\sqrt{2}$ **2.** (a) 13 (b) 5.831 (c) 45 (d) 4.1
3. (a) $\frac{1}{2}\sqrt{14}$ (b) $\frac{2}{3}\sqrt[3]{45}$ (c) $\frac{1}{n}\sqrt{mn}$ (d) $\frac{1}{m+n}\sqrt{m^2-n^2}$ **4.** (a) 0.82 (b) 1.06
(c) 25.35 **5.** (a) $5^{5\frac{1}{5}}$ (b) $(\frac{3}{4})^2$ (c) $8^{\frac{17}{12}}$ (d) $a^{\frac{7}{6}}$ (e) 5^6 (f) $3^{\frac{5}{12}}$ (g) $(\frac{1}{16})^3$ (h) $(\frac{1}{8})^{\frac{3}{4}}$
(i) $(\frac{4}{3})^2$ **6.** (a) 2 (b) 9 (c) 16 (d) 1000 **7.** $h = 10(\frac{3}{4})^n$ **8.** (a) $10^{1.74}$, $10^{.91}$, $10^{-.53}$
(b) 2.8, 18, .22 (c) 3.2 (d) 65 **9.** 4.43×10^2, 5.12×10, 7.6×10^{-2} **10.** $10^{-1.7}$, $10^{-4.35}$, $10^{-2.22}$
11. $10^{.7} \times 10^{-3}$, $10^{.55} \times 10^{-2}$, $10^{.24} \times 10^{-4}$ **12.** $10^{.5378}$, $10^{-1.0655}$, $10^{2.9624}$ **13.** 25.2, 567, 0.168, 0.0760
14. 99,900, 24.5, 9.38

CHAPTER 12

EXERCISE 12·1, Page 216

3. directly **4.** directly **8.** (a) Principal is constant. (b) inverse (c) direct

EXERCISE 12·2, Page 217

1. \$3.84 **2.** \$49.70 **3.** 117, $\frac{117}{365}$ **4.** \$39.80, \$684.80 **5.** \$1503.50

EXERCISE 12·3, Page 219

1. 105 **2.** 217 **3.** 155 **4.** 13 **5.** 33 **6.** 120 **7.** 187 **8.** 364 **9.** \$620
10. \$795.20 **11.** \$26.60

EXERCISE 12·5, Page 221

1. \$192.50 **2.** (a) 5% (b) \$750 (c) \$506.25 (d) $3\frac{3}{4}$ (e) \$1116.83 (f) \$5051.37
3. \$2548.37 **4.** \$720.72

EXERCISE 12·6, Page 222

1. \$2823.16 **2.** \$6298.55 **3.** \$5025 **4.** \$4981 **5.** (a) \$1100, \$1210, \$1331, \$1464.10,
\$1610.51, \$1771.56, \$1948.72, \$2143.59, \$2357.95 \$2593.75 (c) 8 years (d) 10 years

EXERCISE 12·7, Page 225

1. (a) \$608.35 (b) \$772.27 (c) \$1,175.67 **2.** (b) \$12.21 **3.** \$790 **4.** (a) \$647.40
(b) \$931 (c) \$1350 (d) \$2330 (e) \$574 **5.** \$340

EXERCISE 12·8, Pages 227-228

1. 3% per half-year **2.** 3% per quarter **3.** $1\frac{1}{2}$% per month **4.** 4% per half-year
5. $\frac{3}{4}$% per month **6.** 12 **7.** 16 **8.** 174 **9.** 1.17166 **10.** 1.79586 **11.** 1.67535
12. 1.56308 **13.** 1.99650 **14.** 9.86387 **15.** \$7401.20 **16.** \$13,883.69 **17.** \$478.56
18. \$687.78 **19.** \$263.79 **20.** \$743.61 **21.** \$296.90 **22.** \$1070 (slide rule)

EXERCISE 12·9, Pages 229-230

1. \$10,235.77 **2.** \$1220 (slide rule) **3.** yes, $1\frac{1}{2}$% **4.** \$88.36 **5.** \$678.68 **6.** \$574.51
7. \$2104.11 **8.** \$11,837.14 **10.** (a) 6.3 billion (b) By 2004 it will be over 6.8 billion
(c) 24 years

EXERCISE 12·10, Page 231

1. \$456.39 **2.** (a) \$783.53 (b) \$1107.36 (c) \$2653.15 (d) \$6995.40 **3.** \$1856.80
4. \$3322.08 **5.** \$150,000 now

EXERCISE 12·11, Page 232

3. (a) $\$326(1 + \frac{1}{240})^{24}$ (b) $\$756(1 + \frac{1}{120})^{72}$ (c) \$3256
(d) $\$5698(1 + \frac{1}{150})^{66}$ (e) $\$725(1.015)^{17}$

EXERCISE 12·12, Pages 234-235

1. (a) .04, 5 (b) .06, 2 (c) .025, 4 (d) .0075, 12 (e) .02, 20 (f) .03, 5 **5.** (a) .74
(b) .38 (c) .50 (d) .73 **6.** (a) .74 (b) .38 (c) .50 (d) .73 **7.** \$209.86
8. \$568.63 **9.** \$5950.08 **10.** \$2357.71 on her 17th birthday. **11.** \$453.05

REVIEW EXERCISE 12, Pages 235-236

1. (a) directly (b) directly (c) directly **2.** (a) directly (b) inversely (c) inversely
3. \$7418.16 **4.** \$2051.47 **5.** $2\frac{3}{4}\%$ **6.** 37 **7.** 1.41060 **8.** \$876.45 **9.** \$3447.22
10. .005, 36 **12.** \$2.04

CHAPTER 13

EXERCISE 13·1, Page 240

2. (a) $\sin 60°$ (b) $-\cos 20°$ (c) $\sin 10°$ (d) $-\cos 70°$ (e) $-\tan 20°$ (f) $-\cos 30°$
(g) $-\tan 33°$ (h) $\sin 24°$ (i) $-\cos 42°$ (j) $-\cos 17°$ (k) $\sin 25°$ (l) $-\tan 81°$ **3.** (a) .8660
(b) −.7071 (c) .7071 (d) −.8660 (e) .5000 (f) −1.3270 (g) −.7813 (h) −.3443
(i) .9336 (j) −.6293 (k) .2419 (l) −.8090 (m) −.1392 (n) −.8988 (o) −.4663
(p) −1.0724 (q) .1219 (r) −57.2900

EXERCISE 13·2, Page 242

1. $\frac{m}{\sin M} = \frac{n}{\sin N} = \frac{p}{\sin P}$ **2.** (a) .5000 (b) .8660 (c) .9205 (d) .2250
3. $m\angle R = 70°, q = 42.5, r = 40.8$ **4.** $m\angle T = 49°, r = 12.3, t = 13.5$ **5.** $m\angle M = 43°, n = 10.4, m = 37.1$
6. $m\angle E = 28°, e = 44.4, g = 74.6$ **7.** $m\angle R = 60°, c = 61.9, p = 77.5$
8. $m\angle R = 40°, p = 102.3, s = 102.3$ **9.** No solution **10.** $s = \frac{\sin 40° \times r}{\sin 50°}$
11. $M\left(\frac{p}{\cos N} + m, \frac{p}{\sin N}\right)$, $M'\left(\frac{p}{\cos N}, \frac{p}{\sin N}\right)$ (a) $\overrightarrow{PM'} + \overrightarrow{PN} = \overrightarrow{PM}$

EXERCISE 13·3, Page 244

1. $m^2 = p^2 + n^2 - 2pn \cos M$, $s^2 = r^2 + t^2 - 2rt \cos R$, $j^2 = k^2 + h^2 - 2kh \cos J$
2. $r = 3.18, m\angle B = 92°, m\angle M = 56°$ **3.** $a = 5.7, m\angle B = 74°, m\angle T = 41°$
4. (a) $t = 6.5, m\angle P = 87°, m\angle Q = 11°$ (b) $h = 9.9, m\angle J = 84°, m\angle K = 41°$
(c) $a = 10.1, m\angle R = 58°, m\angle M = 49°$ (d) $x = 14.2, m\angle T = 20°, m\angle R = 18°$
(e) $z = 32, m\angle X = 28°, m\angle Y = 20°$ **5.** $k = 9.7, m\angle L = 30°, m\angle M = 25°$
$c = 7.7, m\angle B = 60°, m\angle D = 48°$ $b = 10.2, m\angle Z = 31°, m\angle A = 27°$

EXERCISE 13·4, Page 247

1. 14.6 miles, E32°N **2.** 3.5 miles, 2.3 miles **3.** (a) 31.0 feet (b) 30.7 feet **4.** 129 pounds
5. 214.9 pounds **6.** 908 mph S81°E **7.** 277 mph N48°E

REVIEW EXERCISE 13, Page 248

1. $\sin 45°$ **2.** $-\cos 25°$ **3.** $-\tan 48°$ **4.** .8910 **5.** .3090 **6.** .4067 **7.** −.1736 **8.** .9004
9. −5.6713 **10.** $\frac{c}{\sin 32°} = \frac{7}{\sin 69°}$ **11.** $\frac{p}{\sin 55°} = \frac{22}{\sin 110°}$
12. $c^2 = 14^2 + 15^2 - 2 \times 14 \times 15 \cos 20°$ **13.** $p^2 = 25^2 + 30^2 - 2 \times 25 \times 30 \cos 125°$
14. $c = 3.97, m\angle B = 79°, b = 7.37$ **15.** 5.1 **16.** (a) 73 pounds 32° to 30 pound vector
(b) 2.77 g at 100° to 155 g vector **17.** (a) 994 mph S37°E (b) 845 pounds (c) 357 g

CHAPTER 14

EXERCISE 14·1, Page 254

1. (a) $x^2 + y^2 = r^2$ (b) $(x-3)^2 + (y+4)^2 = 36$ **2.** $(x-5)^2 + (y-6)^2 = 3$ **3.** $(x+2)^2 + (y-7)^2 = 5$
4. $x^2 + y^2 = 16$, (0,0), 4 **5.** (a) (0,0), 2 (b) (1,0), 3 (c) (0, −2), $2\sqrt{2}$ (d) (−3, −5), $2\sqrt{2}$
(e) (3,−7), $2\sqrt{3}$ (f) (−11, 13), 13 (g) (9, 16), $\sqrt{143}$ (h) (−1, 1), $\sqrt{5}$ (i) (−3, −5), $2\sqrt{3}$
(j) (−8, 7), $2\sqrt{6}$ (k) (−4,6), 6

EXERCISE 14·2, Page 255

1. (a) in the interior (b) in the interior (c) in the circle (d) in the circle (e) in the exterior
(f) in the interior **3.** $\{(x,y) | x^2 + y^2 > 121, x,y \in R\}$ **4.** no **5.** no

EXERCISE 14·3, Page 257

1. $m\angle MRN = 90°$, $m\angle AOP = 90°$, $m\angle TVW = 180°$, $m\angle ABC = 180°$, $m\angle DEF = 90°$

EXERCISE 14·4, Pages 260-261

7. (a) $7x + 6y = 85$ (b) $3x - 8y = -73$ (c) $5x + y = 29$ (d) $x + 10y = 64$ (e) $2x - 3y = -4$
8. $\sqrt{27.25}$, 5 **9.** $\sqrt{95}$ **10.** $5x - 12y = 119$ **11.** (a) (0,0), (0,4) (b) undefined (c) 0
(d) perpendicular **12.** (a) $2\sqrt{2}$, $2\sqrt{2}$ (b) (2,2) (c) $2\sqrt{2}$ (d) $2\sqrt{2}$

REVIEW EXERCISE 14, Page 262

1. (a) (0,0), 7 (b) (−2,0), 6 (c) (0,3), 10 (d) (4, −5), 9
2. $x^2 + y^2 = 64$, $(x+2)^2 + y^2 = 25$, $(x+2)^2 + (y+3)^2 = 25$ **3.** All lie in the exterior
4. (a) $\angle PRQ \equiv \angle PJQ$ (b) $\angle TQS \perp \angle SQR$ (c) $m\angle RST = 90°$ (d) $m\angle RQO = 90°$

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